Homework Problems : WEEK V

Problems to be handed in on Monday, May 6 : 8, 1.10.8, 1.10.10 (simply state the inf and sup - no need to prove that they are the inf and sup).

In writing up any of the above, you can use any exercise preceding it and any result/property stated in class without proof.

1. In class we showed the following : $\forall x \geq 0,\ \exists m \in \mathbb{N},\ m-1 \leq x < m$. Write a complete proof of this. Where do you use the condition $x \geq 0$?

2. Generalize the previous exercise in the following way : Show that

$$\forall x \in \mathbb{R},\ \exists m \in \mathbb{Z},\ m-1 \leq x < m.$$

   [Hint: Make two cases : $x \geq 0$ and $x < 0$. For the second case, use the previous exercise with $-x$]

   Recall that a subset $E \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if $\forall a < b,\ \exists x \in E$ such that $x \in (a, b)$.

3. Check that $\mathbb{R}$ is dense in $\mathbb{R}$.

4. Write a complete proof of the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. You may use the previous exercises.

5. Let $E \subseteq \mathbb{R}$ be an arbitrary subset of the reals. Show that E is dense in $\mathbb{R}$ if and only if $\forall x \in \mathbb{R}, \forall \epsilon > 0, \exists y \in E$ such that $y \in (x-\epsilon, x+\epsilon)$. (The 'if' direction was shown in class)

6. Suppose E is dense in $\mathbb{R}$. Let $A \subseteq \mathbb{R}$ be another subset such that $E \subseteq A$. Show that A is also dense in E.

7. If E is dense in $\mathbb{R}$, is $\mathbb{R} \setminus E$ necessarily dense in $\mathbb{R}$? Justify your answer.

8. Show that the set of irrational numbers, i.e., $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. (There are at least two different proofs of this - one of them uses cardinality of sets)

9. If E_1 and E_2 are dense in $\mathbb{R}$, is $E_1 \cap E_2$ also dense in $\mathbb{R}$? Justify.

10. In class, we showed that $|x| - |y| \leq |x-y|$ and claimed that $|y| - |x| \leq |x-y|$. Prove the second inequality. See if you can get the result by simply using previously established results, instead having to go back to definitions. [Hint: $|x-y| = |y-x|$, right ?]

11. Do 1.10.2, 1.10.3, 1.10.4 (ignore the "intepret this geometrically ..." part), 1.10.8 (think induction !), 1.10.10.

12. Do 2.2.6, 2.2.7, 2.2.8, 2.2.9.

13. Do 2.4.1, 2.4.2, 2.4.3, 2.4.4, 2.4.5, 2.4.6 (you can use the formula : $1+2+\ldots+n = n(n+1)/2$).