HW WEEK I Solutions

1. Show: $\forall x \geq 0$, $\exists m \in \mathbb{N}$ s.t. $m-1 \leq x < m$.

By the Archimedean principle,

$\exists N \in \mathbb{N}$ s.t. $x < N$.

Let $S = \{ n \in \mathbb{N} : x < n \}$.

$\therefore S \neq \emptyset$.

Thus S has a least element, say m.

<u>If $m = 1$,</u> then $x < 1$ and also $x \geq 0$.

$\therefore m-1 \leq x < m$

<u>If $m > 1$,</u> then $x < m$ and $m-1 \leq x$ since m is the least element of S, and $m-1 \in \mathbb{N}$.

$\therefore m-1 \leq x < m$.

In both cases, $m-1 \leq x < m$.

[The condition $x \geq 0$ is used in the <u>case $m=1$</u>].

2. We consider 2 cases:

<u>Case 1</u>: $x \geq 0$ This follows from exercise 1.

<u>Case 2</u>: $x < 0$ Consider 2 further cases:

<u>Case 2a</u>: $x \in \mathbb{Z}$

Let $m = x + 1$.

Then $m - 1 \leq x < m$

and $m \in \mathbb{Z}$ since $x \in \mathbb{Z}$

<u>Case 2b</u>: $x \notin \mathbb{Z}$.

Let $y = -x > 0$.

By exercise 1, $\exists\, m \in \mathbb{N}$ s.t. $m - 1 \leq y < m$

$\Rightarrow \quad -(m-1) \geq -y > -m$.

$\Rightarrow \quad -m + 1 \geq x > -m$.

Let $N = -m \in \mathbb{Z}$.

$\therefore \quad N < x \leq N + 1$

Since $x \notin \mathbb{Z} \Rightarrow x \neq N+1$ since $N+1 \notin \mathbb{Z}$.

$\Rightarrow \quad x < N + 1$.

$\Rightarrow \quad N < x < N + 1$

$\Rightarrow \quad N \leq x < N + 1$

and $N \in \mathbb{Z}$. ∎

3. For any (a,b) interval, we pick $\frac{a+b}{2} \in (a,b)$.

Since $\left(\frac{a+b}{2}\right) \in \mathbb{R}$, this shows $\mathbb{R}$ is dense in $\mathbb{R}$.

4. Consider ~~and~~ any interval (a,b).

By the Archimedean principle, $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < b-a$

By exercise 2, $\exists\, m \in \mathbb{Z}$ s.t.

$$m - 1 \leq na < m \quad \left(\text{applying exercise 2 with } x = na\right)$$

Using $m-1 \leq na \Rightarrow \frac{m}{n} - \frac{1}{n} \leq a$

$$\Rightarrow \frac{m}{n} \leq a + \frac{1}{n} < a + (b-a) = b.$$

Also $na < m \Rightarrow a < \frac{m}{n}$.

$$\Rightarrow a < \frac{m}{n} < b \Rightarrow \frac{m}{n} \in (a,b).$$

Since $m \in \mathbb{Z}$, $n \in \mathbb{N}$, $\frac{m}{n} \in \mathbb{Q}$. Done.

5. First show E dense in $\mathbb{R} \Rightarrow \forall x \in \mathbb{R}, \forall \epsilon > 0, \exists y \in E$ s.t. $y \in (x-\epsilon, x+\epsilon)$.

Fix $x \in \mathbb{R}, \epsilon > 0$. Since E is dense, given the interval $(x-\epsilon, x+\epsilon)$ contains $y \in E$. Done.

Next show $\forall x \in \mathbb{R}, \forall \epsilon > 0, \exists y \in E$ s.t. $y \in (x-\epsilon, x+\epsilon)$

$\Rightarrow E$ is dense in $\mathbb{R}$.

Consider any (a,b) interval with $a<b$

Consider $x = \frac{a+b}{2}$ and $\epsilon = \frac{b-a}{2}$.

$\therefore \exists\, y \in E$ s.t. $y \in (x-\epsilon, x+\epsilon)$

$\Rightarrow x - \epsilon < y < x + \epsilon$

$\Rightarrow \frac{a+b}{2} - \left(\frac{b-a}{2}\right) < y < \frac{a+b}{2} + \frac{b-a}{2}$

$\Rightarrow a < y < b$

$\Rightarrow y \in (a,b)$. Done.

6. Consider any (a,b) interval.
Since E is dense, $\exists\, y \in E$ s.t. $y \in (a,b)$.
But $E \subseteq A \Rightarrow y \in A$.
$\therefore y \in A$ and $y \in (a,b)$.
$\Rightarrow A$ is dense in $\mathbb{R}$.

7. $\mathbb{R} \setminus E$ is not necessarily dense:
Let $E = \mathbb{R} \setminus \{0\}$. $\therefore \mathbb{R} \setminus E = \{0\}$.
Clearly E is dense in $\mathbb{R}$, but $\mathbb{R} \setminus E$ is NOT.

8. <u>First proof</u>: Given an interval (a,b),

Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b-a$ [by Archimedean principle]

Consider $y = \frac{a+b}{2}$. If y is irrational,

then we are done since $y \in (a,b)$.

Else, y is rational. ~~Let~~ $x = y + \frac{1}{2\sqrt{2}n}$

We show $x \in (a,b)$.

Since $y > a$ and $x > y \Rightarrow x > a$.

Also $\frac{1}{2\sqrt{2}n} < \frac{1}{2n} \Rightarrow y + \frac{1}{2\sqrt{2}n} < y + \frac{1}{2n} < \frac{a+b}{2} + \frac{b-a}{2}$

$= b$

$\Rightarrow x = y + \frac{1}{2\sqrt{2}n} < b.$

$\Rightarrow a < x < b \Rightarrow x \in (a,b).$

Also x is irrational because otherwise,

$$x - y = \frac{1}{2\sqrt{2}n}$$

$$\Rightarrow \sqrt{2} = \frac{1}{2n(x-y)}$$

the RHS is rational because $x-y$ is rational because x and y are both rational.

9. E_1 and E_2 dense does not mean $E_1 \cap E_2$ dense.

Counterexample: $E_1 = \mathbb{Q}$ $\quad E_2 = \mathbb{R} \setminus \mathbb{Q}$

By previous exercise E_2 is dense.

By exercise 4, E_1 is dense.

However $E_1 \cap E_2 = \phi$ which can never be dense.

10. $|a+b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$.

choose $a = y - x$, $b = x$.

we get $|(y-x)+x| \leq |y-x| + |x|$

$\Rightarrow |y| \leq |y-x| + |x|$.

But $|y-x| = |x-y|$ (since $|a| = |-a| \ \forall a \in \mathbb{R}$).

$\Rightarrow |y| \leq |x-y| + |x|$

$\Rightarrow |y| - |x| \leq |x-y|$. ∎

11. 1.10.2 Consider 2 cases: Case 1: $x \geq y$.

$\Rightarrow x - y \geq 0 \Rightarrow |x-y| = x-y$.

$\Rightarrow \frac{|x-y|}{2} + \frac{x+y}{2} = \frac{x-y}{2} + \frac{x+y}{2} = x$.

Also since $x \geq y$, $\max\{x,y\} = x$.

$\Rightarrow \max\{x,y\} = x = \frac{|x-y|}{2} + \frac{x+y}{2}$

Case 2: $x < y \Rightarrow x - y < 0 \Rightarrow |x-y| = y - x$

$$\therefore \quad \frac{|x-y|}{2} + \frac{x+y}{2} = \frac{y-x}{2} + \frac{x+y}{2} = y.$$

Also $\quad x<y \Rightarrow \max\{x,y\} = y.$

$$\therefore \quad \max\{x,y\} = y = \frac{|x-y|}{2} + \frac{x+y}{2}$$

In both cases, $\max\{x,y\} = \dfrac{|x-y|}{2} + \dfrac{x+y}{2}$ ☒.

1.10.3 $\quad |x-a| < \epsilon$

$\iff -\epsilon < x-a < \epsilon \qquad$ [Property of $|\cdot|$].

$\iff a-\epsilon < x < a+\epsilon \qquad$ [adding a both sides of an inequality].

1.10.4 Since $\quad x<\beta \quad$ and $\quad \alpha<y.$

$\Rightarrow \quad x<\beta \quad$ and $\quad -y<-\alpha.$

Adding the two inequalities:

$$x-y < \beta-\alpha.$$

Similarly $\quad y<\beta \quad$ and $\quad \alpha<x$

$\Rightarrow \quad y<\beta \quad$ and $\quad -x<-\alpha.$

$\Rightarrow \quad y-x < \beta-\alpha.$

$\Rightarrow \quad \max\{x-y,\ y-x\} < \beta-\alpha.$

$\Rightarrow \quad |x-y| < \beta-\alpha.$

1.10.8 We prove by induction on n.

<u>Basis</u>: $n = 1$.

Since $|x_1| \leq |x_1|$ we are done.

<u>Induction Step</u>: Assume $|x_1 + \dots + x_{n-1}| \leq |x_1| + \dots + |x_{n-1}|$

and show $|x_1 + \dots + x_n| \leq |x_1| + \dots + |x_n|$.

Let $a = x_1 + \dots + x_{n-1}$ and $b = x_n$.

Then $|x_1 + \dots + x_n| = |a+b| \leq |a| + |b|$ [Property of $|\cdot|$]

$= |x_1 + \dots + x_{n-1}| + |x_n|$.

By I.H., $|x_1 + \dots + x_{n-1}| \leq |x_1| + \dots + |x_{n-1}|$

$\therefore |x_1 + \dots + x_n| \leq |x_1 + \dots + x_{n-1}| + |x_n| \leq |x_1| + \dots + |x_{n-1}| + |x_n|$

$= |x_1| + \dots + |x_n|$. $\blacksquare$

1.10.10

$$|2x + \pi| < \sqrt{2}$$

$\Rightarrow -\sqrt{2} < 2x + \pi < \sqrt{2}$

$\Rightarrow -\pi - \sqrt{2} < 2x < \sqrt{2} - \pi$

$\Rightarrow \dfrac{-\pi - \sqrt{2}}{2} < x < \dfrac{\sqrt{2} - \pi}{2}$

sup $= \dfrac{\sqrt{2} - \pi}{2}$ inf $= -\dfrac{\pi - \sqrt{2}}{2}$.

B. 2.4.1, 2.4.2 done in class.

2.4.3: We show $\lim_{n\to\infty} s_n = L \Rightarrow \lim_{n\to\infty} (s_n - L) = 0$.

Consider any $\epsilon > 0$.

Since $\lim_{n\to\infty} s_n = L \Rightarrow \exists N \in \mathbb{N}, \forall n \geq N$ $|s_n - L| < \epsilon$.

$\Rightarrow \forall n \geq N \quad |(s_n - L) - 0| < \epsilon$.

$\therefore \lim_{n\to\infty}(s_n - L) = 0$.

Next we show $\lim_{n\to\infty} (s_n - L) = 0 \Rightarrow \lim_{n\to\infty} s_n = L$.

Consider any $\epsilon > 0$.

Since $\lim_{n\to\infty}(s_n - L) = 0 \Rightarrow \exists N \in \mathbb{N}, \forall n \geq N$ $|(s_n - L) - 0| < \epsilon$.

$\Rightarrow \forall n \geq N \quad |s_n - L| < \epsilon$.

So L satisfies definition of $\lim_{n\to\infty} s_n$.

2.4.4 We first show $\lim_{n\to\infty} s_n = L \Rightarrow \lim_{n\to\infty} -s_n = -L$

Consider any $\epsilon > 0$.

Since $\lim_{n\to\infty} s_n = L \Rightarrow \exists N \in \mathbb{N}, \forall n \geq N$ $|s_n - L| < \epsilon$.

Observe that $|s_n - L| = |-s_n - (-L)|$

$\therefore \forall n \geq N, \quad |-s_n - (-L)| < \epsilon$.

$$\Rightarrow \quad -L = \lim_{n\to\infty} -s_n.$$

We now show $\lim_{n\to\infty} -s_n = -L \Rightarrow \lim_{n\to\infty} s_n = L.$

Consider any $\epsilon > 0$.

Since $\lim_{n\to\infty} -s_n = -L$, $\exists N \in \mathbb{N}$, $\forall n \geq N$ $|-s_n-(-L)| < \epsilon$.

But $|-s_n-(-L)| = |s_n - L|$

$\Rightarrow \forall n \geq N$, $|s_n - L| < \epsilon \Rightarrow L = \lim_{n\to\infty} s_n$.

<u>2.4.5</u> We show that

$$\forall m \in \mathbb{N},\ \exists N \in \mathbb{R},\ \forall n \geq N\ |s_n - L| < \frac{1}{m}$$

$$\Rightarrow \forall \epsilon > 0,\ \exists N' \in \mathbb{N},\ \forall n \geq N'\ |s_n - L| < \epsilon.$$

Consider any $\epsilon > 0$. Let m be a natural number s.t. $\frac{1}{m} < \epsilon$. [Archimedean Principle].

$\therefore \exists N \in \mathbb{R}$, $\forall n \geq N$ $|s_n - L| < \frac{1}{m}$.

Choose N' as a natural number larger than N [Archimedean Principle].

$\Rightarrow \forall n \geq N' \geq N$, $|s_n - L| < \frac{1}{m} < \epsilon$.

$\therefore \forall \epsilon > 0$, $\exists N' \in \mathbb{N}$, $\forall n \geq N'$ $|s_n - L| < \epsilon$.

The reverse is easier and is very similar.

2.4.6. $\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2}$

$$= \lim_{n\to\infty} \frac{n(n+1)/2}{n^2} = \lim_{n\to\infty} \frac{1+1/n}{2} = \frac{1}{2}$$

The above reasoning is heuristic. To make formal show

$$\lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2} = \frac{1}{2}.$$

Consider any $\epsilon > 0$.

Let N be a natural number greater than or equal to $\frac{1}{2\epsilon} + 1$

Now, $\forall n \geq N$,

$$\frac{1}{n} \leq \frac{1}{N}$$

$$\Rightarrow \frac{1}{2n} \leq \frac{1}{2N} < \epsilon \qquad \left[\text{since } N \geq \frac{1}{2\epsilon} + 1 \Rightarrow N > \frac{1}{2\epsilon}\right].$$

$$\Rightarrow \left|\frac{n(n+1)}{2n^2} - \frac{1}{2}\right| = \left|\frac{1}{2n}\right| = \frac{1}{2n} < \epsilon.$$

$\blacksquare$.