Alternate proof that irrationals are dense in $\mathbb{R}$.

Consider any interval $(a, b)$.

Consider the set $S = (a, b) \cap \mathbb{Q}$

We show that $(a, b) \setminus S$ is nonempty.

$\mid S \mid \leq \mid \mathbb{Q} \mid = \mid \mathbb{N} \mid \quad \text{since } S \subseteq \mathbb{Q}$

We also know that $\mid \mathbb{Q} \mid < \mid (a, b) \mid$

i.e., there cannot exist any mapping that injective from $(a, b) \rightarrow \mathbb{Q}$

Thus, $(a, b) \neq S$. (Otherwise $f: (a, b) \rightarrow S$

with $f(x) = x$

in injective )\).

$\Rightarrow (a, b) \setminus S \neq \emptyset.$

$\Rightarrow \exists \ y \in (a, b) \setminus S$

Since $S = (a, b) \cap \mathbb{Q}$

This shows $y$ is irrational and $y \in (a, b)$. \(\Box\)