Alternate proof that irrationals are dense in $\mathbb{R}$.

Consider any interval (a,b).

Consider the set $S = (a,b) \cap \mathbb{Q}$

We show that $(a,b) \setminus S$ is nonempty.

$|S| \leq |\mathbb{Q}| = |\mathbb{N}|$ since $S \subseteq \mathbb{Q}$

We also know that $|\mathbb{Q}| < |(a,b)|$
i.e., there cannot exist any mapping that injective from $(a,b) \to \mathbb{Q}$

Thus, $(a,b) \neq S$. (otherwise $f:(a,b) \to S$ with $f(x) = x$ is injective).

$\Rightarrow (a,b) \setminus S \neq \phi$.

$\Rightarrow \exists\, y \in (a,b) \setminus S$

Since $S = (a,b) \cap \mathbb{Q}$
this shows y is irrational and $y \in (a,b)$.

$\blacksquare$