Homework Problems: WEEK III

Problems to be handed in on Monday, April 29: Hand in any three of the following problems.

• 1.6.9 (State the identity that holds between $\text{sup}(A)$ and $\text{inf}(B)$ when $r$ is negative and prove it),

• 5 (Prove only the identity involving the supremum),

• 6 (Prove only the inequality involving the infimum; also state and justify if the inequality can be strict),

• 11, 12.

In any exercise, you can use any exercise preceding it and any result stated in class without proof.

1. Let $E \subseteq \mathbb{R}$ be a nonempty set of real numbers that is bounded. Let $-E = \{-x : x \in E\}$.
   Show that $\text{inf}(-E) = -\text{sup}(E)$, and $\text{sup}(-E) = -\text{inf}(E)$.

2. Let $A \subseteq \mathbb{R}$ be a bounded set of real numbers, and let $r \in \mathbb{R}$ be a fixed real number. Let $B = \{x + r : x \in A\}$. Show that $\text{sup}(A) + r = \text{sup}(B)$ and $\text{inf}(A) + r = \text{inf}(B)$.

3. Do 1.6.9.

4. Let $A \subseteq B$ be two subsets of real numbers. Show that $\text{inf}(A) \geq \text{inf}(B)$ and $\text{sup}(A) \leq \text{sup}(B)$.
   Is it possible for these inequalities to be strict?

5. Let $A, B$ be two subsets of real numbers. Let $C = A \cup B$. Show that $\text{inf}(C) = \min\{\text{inf}(A), \text{inf}(B)\}$ and $\text{sup}(C) = \max\{\text{sup}(A), \text{sup}(B)\}$.

6. Let $A, B$ be two subsets of real numbers. Let $C = A \cap B$. Show that $\text{inf}(C) \geq \max\{\text{inf}(A), \text{inf}(B)\}$ and $\text{sup}(C) \leq \min\{\text{sup}(A), \text{sup}(B)\}$. (You can use any of the previous exercises) Is it possible for these inequalities to be strict?

7. Do 1.6.13 and 1.6.14. The relations will be inequalities - can these inequalities be strict?

8. Let $E \subseteq \mathbb{R}$ be a set of real numbers. Let $L$ be a lower bound for $E$, i.e., $L \leq x \ \forall x \in E$.
   Show that $L = \text{inf}(E)$ if and only if $\forall \epsilon > 0, \exists x \in E$ such that $x < L + \epsilon$.

9. Do 1.6.23 a), c) and d). Justify your answers.

10. Suppose $A, B$ are two subsets of the real numbers such that $A \cap B \neq \emptyset$. Show that $E = \{|a - b| : a \in A, b \in B\}$ has a minimum element.

11. Consider the function $f(x) = 1 - e^{-x}$ and the set $E = \{f(x) : x \geq 0\}$. Find $\text{sup}(E)$ and justify your answer.

12. Consider the function $f(x) = \frac{1}{x}$ and the set $E = \{f(x) : x > 0\}$. Find $\text{inf}(E)$ and justify your answer.

13. Consider the function $f(x) = \frac{1}{x}$ and the set $E = \{f(x) : x < 0\}$. Find $\text{sup}(E)$ and justify your answer. (Note that $x$ ranges over the negative real numbers)

14. Do 1.7.2.