HW 4 MAT 25

1. claim: E is bounded $\iff$ $-E$ is bounded.

Pf: E has a lower bound m. and upper bound M.

$$\Rightarrow \quad m \leq x \leq M \quad \forall x \in E$$

$$\Rightarrow \quad -m \geq -x \geq -M \quad \forall x \in E$$

$$\Rightarrow \quad -m \geq y \geq -M \quad \forall y \in -E.$$

$\Rightarrow$ $-E$ has a lower bound of $-M$ and an upper bound of $-m$.

Therefore $\sup E$, $\inf E$, $\sup(-E)$, $\inf(-E)$ all exist.

Let $M_E = \sup E$, $m_E = \inf E$

$M_{-E} = \sup(-E)$, $m_{-E} = \inf(-E)$

First show $\inf(-E) = -\sup E$

1. Show $\inf(-E) \leq -\sup E$

We will show $-\inf(-E) \geq \sup(E)$

$$m_{-E} \leq x \quad \forall x \in -E \quad [\text{since } m_{-E} = \inf(-E)]$$

$$\Rightarrow \quad -m_{-E} \geq -x \quad \forall x \in -E$$

$$\Rightarrow \quad -m_{-E} \geq y \quad \forall y \in E.$$

$$\Rightarrow \quad -m_{-E} \geq M_E \quad [\text{by def of } M_E = \sup E].$$

Done.

2. Show $\inf(-E) \geq -\sup(E)$.

$$M_E \geq x \quad \forall x \in E \quad [M_E = \sup(E)].$$

$$\Rightarrow -M_E \leq -x \quad \forall x \in E$$

$$\Rightarrow -M_E \leq y \quad \forall y \in -E.$$

$$\Rightarrow -M_E \leq m_{-E} \quad [\text{by def. of } m_{-E} = \inf(-E)].$$

Done.

Next show $\sup(-E) = -\inf(E)$.

1. Show $M_{-E} \leq -m_E$

$$m_E \leq x \quad \forall x \in E \quad [m_E = \inf(E)].$$

$$\Rightarrow -m_E \geq -x \quad \forall x \in E$$

$$\Rightarrow -m_E \geq y \quad \forall y \in -E$$

$$\Rightarrow -m_E \geq M_{-E} \quad [\text{by def of } M_{-E} = \sup(-E)].$$

2. Show $M_{-E} \geq -m_E$

$$M_{-E} \geq x \quad \forall x \in -E \quad [M_{-E} = \sup(-E)]$$

$$\Rightarrow -M_{-E} \leq -x \quad \forall x \in -E$$

$$\Rightarrow -M_E \leq y \quad \forall y \in E$$

$$\Rightarrow -M_E \leq m_E \quad [\text{by def of } m_E = \inf E]$$

$$\Rightarrow M_E \geq -m_E \quad \text{Done.}$$

2: sup relation was shown in class. We show that
$$\inf(A) + r = \inf(B).$$

~~Pf~~ Let $m_A = \inf(A)$ and $m_B = \inf B$

First show $m_A + r \le m_B$

$$m_A \le x \quad \forall x \in A \quad [m_A = \inf A]$$
$$\Rightarrow m_A + r \le x + r \quad \forall x \in A$$
$$\Rightarrow m_A + r \le y \quad \forall y \in B.$$
$$\Rightarrow m_A + r \le m_B \quad [\text{by def. of } m_B = \inf B].$$

Next show $m_A + r \ge m_B$

$$m_B \le x \quad \forall x \in B \quad [m_B = \inf B]$$
$$\Rightarrow m_B - r \le x - r \quad \forall x \in B$$
$$\Rightarrow m_B - r \le y \quad \forall y \in A$$
$$\Rightarrow m_B - r \le m_A \quad [\text{by def. of } m_A = \sup A].$$
$$\Rightarrow m_B \le m_A + r$$

Done.

3. First we show when $r > 0$, $r(\sup A) = \sup B$.

Let $M_A = \sup A$, $M_B = \sup B$.

First show $rM_A \leq M_B$

$$M_B \geq x \quad \forall x \in B \quad [M_B = \sup B]$$

$$\Rightarrow \frac{M_B}{r} \geq \frac{x}{r} \quad \forall x \in B.$$

$$\Rightarrow \frac{M_B}{r} \geq y \quad \forall y \in A$$

$$\Rightarrow \frac{M_B}{r} \geq M_A \quad [\text{by def. of } M_A = \sup A].$$

$$\Rightarrow M_B \geq rM_A.$$

Next show $rM_A \geq M_B$

$$M_A \geq x \quad \forall x \in A \quad [M_A = \sup A].$$

$$\Rightarrow rM_A \geq rx \quad \forall x \in A.$$

$$\Rightarrow rM_A \geq y \quad \forall y \in B.$$

$$\Rightarrow rM_A \geq M_B \quad [\text{by def. of } M_B = \sup B].$$

Next show when $r<0$, $r \sup(A) = \inf(B)$.

Let $r' = -r > 0$.

Let $B' = \{ r'a : a \in A \}$.

$\therefore -B = B'$

By the previous part, $r' \sup A = \sup B'$

(since $r' > 0$).

and from exercise 1, $\sup B' = -\inf(B)$.

$\therefore r' \sup(A) = \sup(B') = -\inf(B)$.

$\Rightarrow -r' \sup(A) = \inf(B)$

$\Rightarrow r \sup(A) = \inf(B)$ $[-r' = r]$.

Done.

4. Let $A \subseteq B$.

If $A = \phi$, $\inf A = \infty$, therefore $\inf(A) \geq \inf(B)$

If B is unbounded below, $\inf B = -\infty$

and $\inf A \geq \inf B$.

So we assume $A \neq \phi$ and B is bounded below.

$\Rightarrow$ A is also bounded below.

and $B \neq \phi$ [since $A \subseteq B$]

$\therefore$ $\inf A$ and $\inf B$ are both real numbers.

Now, $\inf B \leq x \quad \forall x \in B$.

$\Rightarrow \inf B \leq x \quad \forall x \in A$ [$\because A \subseteq B$].

$\Rightarrow \inf B \leq \inf A$. [def. of $\inf A$].

The inequality can be strict:

Let $A = \{2\}$ $\quad B = \{1, 2\}$.

$\inf(A) = 2 > 1 = \inf(B)$.

If $A = \phi$, $\sup A = -\infty$ and so $\sup A \leq \sup B$.

If B is unbounded above, $\sup B = \infty$

and so $\sup A \leq \sup B$.

So we assume

$A \neq \phi$ and B is bounded above.

$\Rightarrow$ $B \neq \phi$ and A is bounded above.

since $A \subseteq B$.

so $\sup A$ and $\sup B$ are real numbers.

$\sup B \geq x \quad \forall x \in B$

$\Rightarrow \sup B \geq x \quad \forall x \in A \quad [\because A \subseteq B]$.

$\Rightarrow \sup B \geq \sup A \quad$ [def. of $\sup A$].

done.

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5. We will show $\sup(C) = \max\{\sup(A), \sup(B)\}$. The relation for inf can be shown in a similar way.

$C = A \cup B$.

By exercise 4, $A \subseteq C \Rightarrow \sup(C) \geq \sup(A)$

and $B \subseteq C \Rightarrow \sup(C) \geq \sup(B)$

$\Rightarrow \sup C \geq \max\{\sup A, \sup B\}$.

So we need to show $\sup(C) \leq \max\{\sup A, \sup B\}$

We consider 2 cases Case 1: $\max\{\sup(A), \sup(B)\} = \sup A$.

If $A = \phi \Rightarrow \sup A = -\infty$

$\Rightarrow \sup B = -\infty$

$\Rightarrow B = \phi$.

$\Rightarrow \quad C = A \cup B = \phi$.

$\Rightarrow \sup C = -\infty$.

$\therefore \quad \sup C = \sup A = \sup B = -\infty$.

Done.

If A is unbounded above,

$C = A \cup B$ is also unbounded above.

$\Rightarrow \quad \sup C = \sup A = \infty$. Done.

So assume $A \neq \phi$ and A is bounded above.

$\Rightarrow \sup A$ is a real number.

$\Rightarrow \sup B (\leq \sup A)$ is a real number.

Now, $\sup A \geq x \quad \forall x \in A$.

~~Also~~ $\sup A \geq \sup B \geq x \quad \forall x \in B$.

$\therefore \quad \sup A \geq x \quad \forall x \in A \cup B$.

$\Rightarrow \sup A \geq x \quad \forall x \in C$.

$\Rightarrow \sup A \geq \sup C \quad$ [def. of $\sup C$].

Case 2: $\max \{\sup(A), \sup(B)\} = \sup B$.

Proof is similar to Case 1., with A + B interchanged.

6. We show the relation for ~~inf~~ infimum. The supremum proof is similar.

Show $\inf(C) \geq \max\{\inf(A), \inf(B)\}$.

$C = A \cap B$.

By exercise 4,

$\Rightarrow C \subseteq A \Rightarrow \inf C \geq \inf A$.

and $C \subseteq B \Rightarrow \inf C \geq \inf B$

$\Rightarrow \inf C \geq \max\{\inf A, \inf B\}$.

The inequality can be strict:

e.g. $A = \{1, 2, 3\}$ $B = \{0, 2, 3\}$.

$C = \{2, 3\}$.

$\inf C = 2 > 1 = \max\{1, 0\} = \max\{\inf A, \inf B\}$.

7. Claim: $\sup C = \sup A + \sup B$.

Pf: $\sup A \geq x \quad \forall x \in A$.

and $\sup B \geq y \quad \forall y \in B$.

$\Rightarrow \sup A + \sup B \geq x + y \quad \forall x \in A, y \in B$

$\Rightarrow \sup A + \sup B \geq z \quad \forall z \in C$.

$\therefore$ $\sup A + \sup B$ is an upper bound on C.

We now show that $\forall \epsilon > 0, \exists z \in C$ s.t.

$$z > \sup A + \sup B - \epsilon$$

This will prove $\sup A + \sup B = \sup C$.

<u>Consider any $\epsilon > 0$.</u>

We know that for $\epsilon/2$, $\exists a \in A$ s.t.

$$a > \sup A - \epsilon/2$$

and $\exists b \in B$ s.t. $b > \sup B - \epsilon/2$.

$$\Rightarrow \quad a + b > \sup A + \sup B - \epsilon.$$

Thus $z = a + b \in C$ and $z > \sup A + \sup B - \epsilon$.

<u>Done</u>

The proof for the relation

$$\inf C = \inf A + \inf B \text{ is}$$

similar and is omitted.

8. First show $L = \inf E \implies \forall \epsilon > 0, \exists x \in E$ s.t. $x < L + \epsilon$.

~~Pf by contrapositive~~ Proof by contradiction:

Suppose NOT$(\forall \epsilon > 0, \exists x \in E$ s.t. $x < L+\epsilon)$.

$\implies \exists \epsilon > 0, \forall x \in E$ s.t. $L + \epsilon \leq x$.

But by def. of $L = \inf E$, this implies

$$L + \epsilon \leq L$$

which is a contradiction since $\epsilon > 0$.

Next show that

$\forall \epsilon > 0, \exists x \in E$ s.t. $x < L + \epsilon \implies L = \inf E$.

Since L is a lower bound, $x \leq L \; \forall x \in E$.

So need to show that

$$\forall L' \in \mathbb{R}, \; (x \geq L' \; \forall x \in E) \implies (L \geq L').$$

By contrapositive: $L < L' \implies \exists x \in E, \; x < L'$.

Let $\epsilon = L' - L > 0$.

By hypothesis, for this $\epsilon > 0$, $\exists x \in E$, s.t. $x < L + \epsilon$.

$\implies \exists x \in E$ s.t. $x < L'$

$\boxtimes$

9. 1.6.23 a) $A = \mathbb{N}$, $B = \mathbb{R} \setminus \mathbb{N}$.

$\delta(A,B) = 0$.

i.e., $0 = \inf \{|a-b| : a \in A, b \in B\}$.

since $|a-b| \geq 0$ $\forall a \in A, b \in B$.

0 is a lower bound.

By exercise 8, it suffices to show

$\forall \epsilon > 0$, $\exists x \in \{|a-b| : a \in A, b \in B\}$ s.t. $x < 0 + \epsilon$

Given $\epsilon > 0$,

If ϵ is an even natural number.

choose $a = 1 \in A$, $b = 1.5 \in B$

set $x = |a-b| = 0.5 < 1 < \epsilon$.

If ϵ is NOT an even natural number.

choose $a = 1 \in A$, $b = 1 + \epsilon/2 \in B$ (since $\epsilon/2 \notin \mathbb{N}$)

set $x = |a-b| = \frac{\epsilon}{2} < \epsilon$.

1.6.23 b) $A = \{x\}$, $B = [0,1]$.

claim: $0 = \inf \{|a-b| : a \in A, b \in B\} \iff x \in [0,1]$.

Pf: If $x \in [0,1]$, then choose $a = x$, $b = x$

and therefore $|a-b| = 0$.

Thus $0 =$ minimum of $\{|a-b| : a \in A, b \in B\}$.

Thus $x \in [0,1] \Rightarrow 0 = \inf \{|a-b| : a \in A, b \in B\}$.

~~If $x \notin [0,1]$ we have 2 cases:~~

We now show

$$0 = \inf \{|a-b| : a \in A,\ b \in B\} \Rightarrow x \in [0,1].$$

Pf by contrapositive : $x \notin [0,1]$. Consider 2 cases :

$\underline{x > 1}$: Let $\epsilon = x - 1 > 0$

$$\Rightarrow x = 1 + \epsilon \geq y + \epsilon \quad \forall y \in B.$$

since $y \leq 1 \ \forall y \in B$.

$$\Rightarrow x - y \geq \epsilon \quad \forall y \in B$$

$$\Rightarrow |x-y| \geq \epsilon \quad \forall y \in B.$$

But then ϵ is a lower bound and $\epsilon > 0$

$\Rightarrow$ 0 cannot be infimum.

$\underline{1.6.23c)}$ $A = \{x\}$ $B = (0,1)$.

$\underline{\text{Claim}}$: $0 = \inf \{|a-b| : a \in A,\ b \in B\} \Longleftrightarrow x \in [0,1]$

$\underline{\text{Pf}}$: If $x \in [0,1]$, we show $0 = \inf \{|a-b| : a \in A, b \in B\}$

0 is a lower bound, so we show

$$\forall \epsilon > 0,\ \exists y \in \{|a-b| : a \in A, b \in B\} \text{ s.t. } y < 0 + \epsilon.$$

~~Given $\epsilon > 0$,~~

~~If $x > \epsilon/2$, choose $a = x$, $b = x - \epsilon/2 > 0$~~

~~Since $x \leq 1$ $b = x - \epsilon/2 < 1$, $\therefore b \in B$.~~

~~Set $y = |a-b| = \epsilon/2 < \epsilon$~~

~~Given $\epsilon > 0$,~~

~~let $\delta = \min$~~

Consider any $\epsilon > 0$.

If $x \in (0,1)$ choose $a = x$, $b = x \in B$

set $y = |a-b| = 0 < \epsilon$ Done.

If $x = 0$, choose $a = x$, $b = \min\{\frac{1}{2}, \frac{\epsilon}{2}\}$

Since $\frac{\epsilon}{2} > 0 \Rightarrow b > 0$.

and $b \leq \frac{1}{2} \Rightarrow b < 1$.

$\Rightarrow b \in (0,1) = B$.

set $y = |a-b| = |0-b| = b = \min\{\frac{1}{2}, \frac{\epsilon}{2}\} \leq \frac{\epsilon}{2} < \epsilon$.

Done.

If $x = 1$, choose $a = x$, $b = \max\{\frac{1}{2}, 1-\frac{\epsilon}{2}\}$

Since $b \geq \frac{1}{2} > 0 \Rightarrow b > 0$.

since $1-\frac{\epsilon}{2} < 1$ and $\frac{1}{2} < 1$

$\Rightarrow b < 1$.

$\Rightarrow b \in (0,1) = B$.

set $y = |a-b| = |1-b| \leq 1-(1-\frac{\epsilon}{2})$ (since $b \geq 1-\frac{\epsilon}{2}$

$= \frac{\epsilon}{2} < \epsilon$

$\blacksquare$

10. $0 = \min E$. ~~So~~ Let $x \in A \cap B \neq \phi$.

choose $\underset{\in A}{a = x}$, $\underset{\in B}{b = x}$.

$\Rightarrow |a-b| = 0$.

Thus $0 \in E$ and $\therefore$ $0 = \min E$

since $0 \leq x \ \forall x \in E$.

11. $\sup E = 1$.

First, $e^{-x} \geq 0 \quad \forall x \in \mathbb{R}$.

$\Rightarrow -e^{-x} \leq 0 \quad \forall x \in \mathbb{R}$.

$\Rightarrow 1 - e^{-x} \leq 1 \quad \forall x \in \mathbb{R}$.

$\Rightarrow f(x) \leq 1 \quad \forall x \in \mathbb{R}$.

$\therefore$ 1 is an upper bound for E.

Second, we show $\forall \epsilon > 0, \exists y \in E$ s.t. $1 - \epsilon < y$.

If $\epsilon \geq 2$. choose $y = 1 - e^{-0} = 0$

$1 - \epsilon \leq -1$ since $\epsilon \geq 2$.

$\Rightarrow 1 - \epsilon \leq -1 < 0 = y$. ✓.

If $\epsilon < 2$ choose $x = -\ln(\epsilon/2)$

since $\epsilon < 2 \Rightarrow x = -\ln(\epsilon/2) > 0$

Now $y = f(x) = e^{-(-\ln \epsilon/2)}$

$= \epsilon/2 < \epsilon$ <u>Done.</u>

12. <u>$\inf E = 0.$</u>

First $\frac{1}{x} > 0 \quad \forall x > 0.$

$\Rightarrow$ 0 is a lower bound.

~~Second~~ Now we show $\forall \epsilon > 0, \exists y \in E, \; y < 0 + \epsilon$
$(y < \epsilon).$

For any $\epsilon > 0$, choose $x = 2/\epsilon > 0.$

set $y = f(x) = \frac{1}{x} = \epsilon/2 < \epsilon$ <u>Done.</u>

13. <u>$\sup E = 0.$</u>

First $\frac{1}{x} < 0 \quad \forall x < 0.$

$\Rightarrow$ 0 is an upper bound.

Now we show $\forall \epsilon > 0, \exists y \in E, \; 0 - \epsilon < y.$
$(-\epsilon < y).$

Given $\epsilon > 0$, choose $x = -2/\epsilon < 0$ since $\epsilon > 0.$

set $y = f(x) = \frac{1}{x} = -\epsilon/2 > -\epsilon$ <u>Done.</u>

1.7.2 Hypothesis : $\forall x > 0, \exists n \in \mathbb{N}, \frac{1}{n} < x$.

To prove: $\mathbb{N}$ has no upper bound.

Formally, need to show.

$$\forall y \in \mathbb{R}, \exists n \in \mathbb{N}, y < n.$$

Given $y \in \mathbb{R}$, ~~set x = 1/y~~

If $y \leq 0$ then $n = 1$ works.

If $y > 0$, set $x = 1/y$.

By hypothesis, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$

$$\Rightarrow \frac{1}{x} < n.$$

$$\Rightarrow y = \frac{1}{x} < n.$$

Done.