HW 3 MAT 25

1. Field with $\overline{2}=0$:

$X = \{0, 1\}$.

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Here $\overline{2} = 1+1 = 0$

Field with $\overline{3}=0$:

$X = \{0, 1, 2\}$.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(addition modulo 3)

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

(multiplication modulo 3)

Here $\overline{3} = 1+1+1 = 0$

2. Proof by induction on m.

$P(m) \equiv \overline{mn} = 0 \quad \forall\, m \geq 1$.

1. Establish $P(1)$: $\overline{(1)(n)} = \overline{n} = 0$ by assumption

2. Assume $P(m-1)$ true.

Consider $\overline{mn} = \underbrace{1+1+1+1+ \dots +1}_{mn \text{ times}}$

$= \underbrace{1+1+1 \dots +1}_{(m-1)n \text{ times}} + \underbrace{1+1+\dots 1}_{n \text{ times}}$

$= \overline{(m-1)n} + \overline{n}$

$= 0 + \bar{n}$ (By Induction Hypothesis).

$= \bar{n}$ (By Field axiom 3).

$= 0$ (By assumption).

Thus $P(m)$ is true. $\blacksquare$

3. $a < b$ and $c < d$.

Since $c < d$

$\Rightarrow c + (-c) < d + (-c)$ [By order axiom 2]

$\Rightarrow 0 < d + (-c)$ [Field axiom 4].

$\Rightarrow a \cdot (d + (-c)) < b(d + (-c))$ [Order axiom 4]

~~$\Rightarrow ad + a(-c) < bd + b(-c)$ [Field axiom 9]~~

$\Rightarrow (d + (-c)) \cdot a < (d + (-c)) \cdot b$ [Field axiom 5]

$\Rightarrow d \cdot a + (-c) \cdot a < d \cdot b + (-c) b$ [Field axiom 9].

$\Rightarrow a \cdot d + (-ca) < bd + (-cb)$ [Field axioms 5 and consequence that $(-a)b = -(ab)$]

Add $cb + ca$ both sides by order axiom 3.

$\Rightarrow ad + (-ca) + (cb + ca) < bd + (-cb) + (cb + ca)$

$\Rightarrow ad + (cb + ca) + (-ca) < bd + (-cb + cb) + ca$

$\Rightarrow ad + cb + 0 < bd + 0 + ca$ [Field axiom 4]

$\Rightarrow ad + cb < bd + ca$ [Field axiom 3].

$\Rightarrow ad + bc < ac + bd$ [Field axioms 1 + 5]

4. Since $x \neq 0$, we consider 2 exhaustive cases by Order axiom 1:

<u>Case 1</u>: $0 < x \Rightarrow 0 \cdot x < x \cdot x$ [by order axiom 4].

$\Rightarrow 0 < x^2$ [By consequence that $0 \cdot x = 0$]

<u>Case 2</u>: $x < 0 \Rightarrow 0 < -x$ [By consequence that $x < 0 \Leftrightarrow -x > 0$]

$\Rightarrow x(-x) < 0 \cdot (-x)$ [Order axiom 4]

$\Rightarrow x(-x) < 0$ [By consequence $0 \cdot x = 0$]

$\Rightarrow -(x)(x) < 0$ [By consequence $(-a)(b) = -(a \cdot b)$]

$\Rightarrow -x^2 < 0$

$\Rightarrow x^2 > 0$ [By consequence $a < 0 \Leftrightarrow -a > 0$ and $-(-a) = a$]

In both cases, we get $0 < x^2$ ✓

5. Since $1 = (1)(1) = 1^2$ and $1 \neq 0$.

By exercise 4, $0 < 1$.

6. We first establish (ii): $\overline{n+m} = \overline{n} + \overline{m} \quad \forall\, n, m \in \mathbb{N}$.

$$\overline{n+m} = \underbrace{1 + 1 + \dots + 1}_{n+m \text{ times}}$$

$$= \underbrace{1 + 1 + \dots 1}_{n \text{ times}} + \underbrace{1 + 1 + \dots + 1}_{m \text{ times}}.$$

$$= \overline{n} + \overline{m}$$

Now establish (i): $n \neq m \Rightarrow \overline{n} \neq \overline{m}$.

First we show that if $a \in \mathbb{N}$ then $\overline{a} > 0$.

Pf: By induction on a.

For $a = 1$. $\Rightarrow \overline{a} = 1 > 0$ by exercise 5.

For $a \geq 2$, assume $\overline{a-1} > 0$ and show $\overline{a} > 0$.

$$\overline{a} = \underbrace{1 + 1 + \dots + 1}_{a \text{ times}}$$

$$= \underbrace{1 + 1 + \dots + 1}_{a-1 \text{ times}} + 1.$$

So $\bar{a} = \overline{a-1} + 1$.

By induction hypothesis, $\overline{a-1} > 0$.

Thus $\overline{a-1} + 1 > 0+1$ [By order axiom 3]

$\Rightarrow \bar{a} > 1$ [Field axiom 3].

But exercise 5 shows $1 > 0$.

$\therefore \bar{a} > 0$ [Order axiom 2]. $\blacksquare$

Now to show $n \neq m \Rightarrow \bar{n} \neq \bar{m}$

Assume $m < n$ (the other case is similar).

$\Rightarrow n-m > 0 \Rightarrow \overline{n-m} > 0$ [by what we showed before]

Now by part (ii)

$$\bar{n} = \overline{(n-m)+m} = \overline{n-m} + \bar{m}$$

Now take $\overline{n-m} > 0$

and add $\bar{m}$ both sides by order axiom 3.

$\Rightarrow \overline{n-m} + \bar{m} > 0 + \bar{m}$

$\Rightarrow \bar{n} > \bar{m}$ [By $\bar{n} = \overline{n-m} + \bar{m}$ and field axiom 3]

$\Rightarrow \bar{n} \neq \bar{m}$ by order axiom 1.

6(iv) In the proof of part (i) we showed . . .

$$\bar{a} > 0 \quad \forall \quad a \in \mathbb{N}$$

We now show if $a \neq 1$ and $a \in \mathbb{N}$, $\bar{a} > 1$.

Since $a \neq 1$, and $a \in \mathbb{N}$

$$a - 1 \in \mathbb{N}$$

$$\Rightarrow \overline{a-1} > 0$$

$$\Rightarrow \overline{a-1} + 1 > 0 + 1 \qquad \text{[Order axiom 3]}$$

$$\Rightarrow \bar{a} > 1 \qquad \text{[Field axiom 3]}.$$

$\therefore$ either $a = 1$, and so $\bar{a} = 1$.

or $a \neq 1$, and $\bar{a} > 1$.

Now if $n \in \mathbb{N}$ and $n = 1$.

Then $(\bar{n})^2 = 1^2 = 1 = \bar{n}$ so $(\bar{n})^2 \geq \bar{n}$.

If $n \in \mathbb{N}$ and $n > 1$.

Then $\bar{n} > 1$

and also $\bar{n} > 0$.

$$\Rightarrow (\bar{n})(\bar{n}) > (1)(\bar{n}) \qquad \text{[Order axiom 4]}$$

$$\Rightarrow (\bar{n})^2 > \bar{n}$$

$$\therefore (\bar{n})^2 \geq \bar{n} \quad \forall \; n \in \mathbb{N}.$$

7. Since $0 < 1$

$\Rightarrow -1 < 0$ [consequence $a > 0 \iff -a < 0$]

But exercise 4 shows $x^2 > 0$ if $x \neq 0$.

since $-1 < 0$ this shows for $x \neq 0$, x^2 cannot be -1.

And if $x = 0$, then $x^2 = 0 \neq -1$.

Thus, in no ordered field X can there exist an element $x \in X$ such that $x^2 = -1$.

8. I will present the proof suggested by a student in class:

By induction on $|E|$.

Case when $|E| = 1$: $E = \{x\}$ and clearly x is a maximum and minimum.

<u>Case when $|E| \geq 2$</u>: Assume the statement is true for all sets E such that $|E| = n-1$ and show it holds when $|E| = n$.

Consider any $x \in E$.

Let $E' = E \setminus \{x\}$.

$|E'| = |E| - 1 = n-1$.

Thus, E' has a max and min.

Say $M' = $ max for E'

and $m' = $ min for E'.

If $x \leq M'$ then ~~M' is a max for~~

let $M = M'$.

else let $M = x$.

If $x \leq m'$, let $m = x$; else let $m = m'$.

<u>Claim</u>: M is a max for E, and m is a min for E.

First show M is a max for E.

Pf: <u>Case 1</u>: $x \leq M'$. So $M = M'$

$\forall y \in E$, if $y \neq x$, then $y \in E'$

$\Rightarrow y \leq M'$ [since M' is a max for E'].

and if $y = x$ then $y \leq M'$ since $x \leq M'$.

$\Rightarrow \forall y \in E$, $y \leq M' = M$

Also $M = M' \in E' \subseteq E \Rightarrow M \in E$.

Thus M is a max for E.

This argument goes after Case 2

~~Next show~~

Case 2: $M' < x$. So $M = x$.

$\forall y \in E$, if $y \neq x$, then $y \in E'$

$\Rightarrow y \leq M' < x$

$\Rightarrow y < x = M$.

If $y = x$, then $y = x = M$.

$\therefore \forall y \in E, \quad y \leq M$.

The argument for $m = \min$ for E is similar.

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9. If X is a finite set, it has a maximum say $M \in X$ by exercise 8.

We know $0 < 1$ from exercise 5.

Add M both sides:

$0 + M < 1 + M$.

$\Rightarrow M < 1 + M$.

But this means the element $1 + M > M$ which is a contradiction to M being the maximum for X.

6(iii) Proof by induction on m.

For $m=1$: $\overline{(1)(n)} = \overline{n} = 1\cdot\overline{n} = (\overline{1})(\overline{n})$.

For $m \geq 2$: Assume $\overline{(m-1)n} = (\overline{m-1})(\overline{n})$ and show $\overline{mn} = (\overline{m})(\overline{n})$.

$$
\begin{aligned}
\overline{mn} &= \overline{(m-1)n + n} \\
&= \overline{(m-1)n} + \overline{n} && \text{by part (ii)} \\
&= (\overline{m-1})(\overline{n}) + \overline{n} && \text{by induction hypothesis} \\
&= (\overline{m-1})(\overline{n}) + 1\cdot\overline{n} && [\text{Field axiom } 7] \\
&= (\overline{m-1} + 1)(\overline{n}) && [\text{Field axiom } 9]. \\
&= (\underbrace{1+1+\dots+1}_{m-1 \text{ times}} + 1)(\overline{n}) \\
&= (\underbrace{1+1+\dots 1}_{m \text{ times}})(\overline{n}) \\
&= (\overline{m})(\overline{n}) && ✓
\end{aligned}
$$

10. $a < b$

$\Rightarrow ac < bc$ [by order axiom 4 since $c > 0$.]

Also $c < d$

$\Rightarrow bc < bd$ [by order axiom 4 since $b > 0$]

Since $ac < bc$ and $bc < bd$

$\therefore ac < bd$ [by order axiom 2]

11. $a > 0$ and $b > 0$

$\Rightarrow a \cdot b > 0 \cdot b$ [order axiom 4].

$\Rightarrow a \cdot b > 0$ [consequence $0 \cdot b = 0$].

$a < 0$ and $b > 0$

$\Rightarrow a \cdot b < 0 \cdot b$ [order axiom 4].

$\Rightarrow ab < 0$ [consequence $0 \cdot b = 0$].

$a < 0$ and $b < 0$

$\Rightarrow -a > 0$ and $b < 0$ [consequence $a < 0 \Leftrightarrow -a > 0$]

$\Rightarrow (-a)(b) < (-a)(0)$ [Order axiom 4].

$\Rightarrow -(ab) < 0$ [Consequence $0 \cdot a = 0$ and $(-a)(b) = -(ab)$.

$\Rightarrow ab > 0$ [Consequence $a < 0 \Leftrightarrow -a > 0$ and $-(-a) = 0$

12. $a \leq b \Rightarrow a^2 \leq b^2$

follows from exercise 10 using $\begin{bmatrix} a = a \\ b = b \\ c = a \\ d = b \end{bmatrix}$

In the other direction

$a^2 < b^2$

$\Rightarrow a^2 + (-a^2) < b^2 + (-a^2)$ [order axiom 3]

$\Rightarrow 0 < b^2 + (-a^2)$ [Field axiom 3].

$\Rightarrow 0 < (b+a)(b+(-a))$ [Consequence from class]

Now since $a > 0$ and $b > 0$

$\Rightarrow a + b > b + 0$ and $b > 0$ [order axiom 3]

$\Rightarrow a + b > 0$. [order axiom 2]

Consider 3 cases by order axiom 1:

Case 1: $b + (-a) = 0$

$\Rightarrow (b+(-a))(b+a) = 0$ [consequence $0 \cdot a = 0$]

this contradicts $0 < (b+a)(b+(-a))$.

Case 2: $b + (-a) < 0$

but then $(b+(-a))(b+a) < 0(b+a)$ [since $a+b > 0$ and order axiom 4]

$\Rightarrow \quad (b+(-a))(b+a) < 0$

this contradicts $0 < (b+a)(b+(-a))$

Thus $b+(-a) > 0$

$\Rightarrow b+(-a)+a > 0+a$

$\Rightarrow \quad b+0 > a$

$\Rightarrow \quad b > a.$ $\blacksquare$

13. Let M be the maximum for E.

$\Rightarrow M \in E$ and $\forall x \in E, \; x \leq M$.

We claim M is also the least upper bound.

Since $\forall x \in E, \; x \leq M$, M is an U.B.

Consider any $M' \in X$ such that M' is an U.B. for E.

$\Rightarrow \quad \forall x \in E, \quad x \leq M'$

Since $M \in E$, this shows $M \leq M'$.

Thus $M \leq M'$ for any U.B. M' for E.

$\Rightarrow$ M is the least upper bound.

14. 1.6.1 from text

E is bounded $\Rightarrow$ E has an upper bound M and a lower bound m.

~~Consider~~ Let $r' =$ maximum of $|M|$ and $|m|$.

Claim: We show $\forall x \in E$, $|x| \leq r'$.

Pf: $\forall x \in E$

$$x < M \leq |M| \leq r'$$

and $m < x$

$$\Rightarrow -x < -m \leq |m| \leq r'$$

$$\Rightarrow x \leq r' \text{ and } -x \leq r'$$

$$\Rightarrow |x| \leq r'. \quad \blacksquare$$

Set $r = r' + 1$.

From the claim $\forall x \in E$, $|x| \leq r'$

and $r' < r' + 1 = r$.

$$\Rightarrow |x| \leq r' \quad \forall x \in E. \quad \blacksquare$$

	Max	Min	Least U.B.	Greatest L.B.
15 (i)	7	2	7	2
(ii)	none	none	5	-5
(iii)	none	none	5	-5
(iv)	5	-5	5	-5
(v)	5	-5	5	-5
(vi)	none	none	none	0
(vii)	none	none	$\sqrt{2}$	0