Homework Problems : WEEK II

Problems to be handed in on Monday, April 15 : A.2.1 (e), A.2.3 (a) (Justify your description for A.2.3 (a)), Problem 7 (iii), Problem 12 (ii) and (iii) (You can assume the other parts of Problem 12)

1. Do A.2.1, A.2.2, A.2.3.

2. Let A be a set, and B_i, $i \in I$ be a family of sets indexed by I. Prove De Morgan's laws :

   (i) $A \setminus (\bigcup_{i\in I} B_i) = \bigcap_{i\in I}(A \setminus B_i)$
   (ii) $A \setminus (\bigcap_{i\in I} B_i) = \bigcup_{i\in I}(A \setminus B_i)$

3. Let A, B be finite sets. Assume that $|A \cup B| = |A| + |B|$ when $A \cap B = \emptyset$. Write a complete proof of the formula $|A \cup B| = |A| + |B| - |A \cap B|$ for general finite sets A, B.

4. Argue carefully that the cardinality of the set $(0, 1)$ is the same as the cardinality of the intervals (a, b) and $[a, b]$ for any real numbers $a \neq b$.

5. Show that $A \subseteq B$ implies $|A| \leq |B|$.

6. Let A, B, C be sets. Show that if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

7. **Countable union of countable sets** Let I be a countable set. Let $A_i, i \in I$ be a family of sets such that each A_i is countable. We will show that $\bigcup_{i\in I} A_i$ is countable.

   (i) Show that there exists a family of sets $C_1, C_2, C_3, \ldots$, i.e., a family of sets C_i indexed by $i \in \mathbb{N}$ such that C_i is countable for every $i \in \mathbb{N}$ and $\bigcup_{i\in I} A_i = \bigcup_{i\in\mathbb{N}} C_i$.
   (Hint : Some of the C_i can be empty sets.)
   (ii) Show that there exists a family of sets $B_i, i \in \mathbb{N}$ such that $\bigcup_{i\in\mathbb{N}} C_i = \bigcup_{i\in\mathbb{N}} B_i$, each B_i is countable and $B_i \cap B_j = \emptyset$ for any $i \neq j$, i.e., the B_i's are pairwise disjoint.
   (Hint : Think of the construction $B_i = C_i \setminus (C_1 \cup C_2 \cup \ldots \cup C_{i-1})$.)
   (iii) Show that $\bigcup_{i\in\mathbb{N}} B_i$ is countable for the family of sets $B_i, i \in \mathbb{N}$ from part (ii). You may assume that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.
   (iv) Hence conclude $\bigcup_{i\in I} A_i$ is countable.

8. Consider the set of all functions $f : \mathbb{N} \to \{0, 1\}$, i.e., the functions map the natural numbers to 0 or 1. Do you think the set of all such functions is countable or uncountable ?

9. Show that the set $\mathbb{Q}$ of rational numbers is countable. (Hint : Use Exercise 7 and the fact that $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$).

10. Recall from class we showed that $[0, 1]$ is not countable. Use this fact and exercise 9 to show that $|\mathbb{Q}| \neq |\mathbb{R}|$. (Of course, since $\mathbb{Q} \subset \mathbb{R}$, exercise 5 shows that $|\mathbb{Q}| \leq |\mathbb{R}|$.)

11. Do 1.3.7 on page 7.

12. Show the following hold in any set X with $+, \cdot$ that satisfy the 9 field axioms. You can use without proof Consequence 1 shown in class, i.e., the fact that $0 \cdot a = 0$ for all $a \in X$.

    (i) $(a + b)^2 = a^2 + ab + ab + b^2$.
    (ii) $(-a) \cdot (b) = -(a \cdot b)$. (Hint: Start with $(a + (-a)) \cdot b$)
    (iii) Show that for every $a \in X$, there is a unique $(-a)$. That is, show that if there exist $a' \in X$ and $a'' \in X$ such that $a + a' = a' + a = 0$, and $a + a'' = a'' + a = 0$, then $a' = a''$. (Hint: Start with $(a' + a) + a''$.) Hence, show that $-(-a) = a$.
    (iv) $(-1)(-1) = 1$. (Hint : Use part (ii) and (iii) above)