HW WEEK2 MAT 25

A.2.1 a) Show $A \cup B = B \iff A \subset B$.

First show $A \cup B = B \Rightarrow A \subset B$

Need to show $x \in A \Rightarrow x \in B$.

But $x \in A$

$\Rightarrow x \in A \cup B = B$. [By assumption]

$\Rightarrow x \in B$. Done.

Next show $A \subset B \Rightarrow A \cup B = B$.

Show first $A \cup B \subseteq B$

Consider $x \in A \cup B$

Case 1: $x \in A$

$\Rightarrow x \in B$ (since $A \subset B$)

Case 2: $x \in B$. Done

In both cases, $x \in B$.

$\therefore x \in A \cup B \Rightarrow x \in B$.

Next show $B \subseteq A \cup B$

Consider $x \in B$

$\Rightarrow x \in A \cup B$. Done.

A.2.1. b) Show $A \cap B = A \iff A \subset B$.

First show $A \cap B = A \implies A \subset B$.

Need to show $x \in A \implies x \in B$.

But $x \in A$

$\implies x \in A \cap B$ [Since $A = A \cap B$ by assumption].

$\implies x \in B$.

Done

Next show $A \subset B \implies A \cap B = A$.

First show $A \cap B \subseteq A$

Consider $x \in A \cap B$

$\implies x \in A$ Done.

Next show $A \subseteq A \cap B$.

Consider $x \in A$

$\implies x \in B$ (Since $A \subset B$ by assumption).

$\implies x \in A \cap B$ Done.

A.2.1 c) Show $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

First show ~~$A \cap B$~~ $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$

Consider $x \in (A \cup B) \cap C$

$\implies x \in A \cup B$ and $x \in C$.

$\implies x \in A$ or $x \in B$, and $x \in C$.

Case 1: $x \in A$

$\implies x \in A$ and $x \in C$.

$\implies x \in A \cap C$.

$\Rightarrow x \in (A \cap C) \cup (B \cap C)$.

<u>Case 2:</u> $x \in B$

$\Rightarrow x \in B$ and $x \in C$

$\Rightarrow x \in B \cap C$

$\Rightarrow x \in (A \cap C) \cup (B \cap C)$

In both cases, $x \in (A \cap C) \cup (B \cap C)$, so we are done.

<u>Next show $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$</u>

Consider $x \in (A \cap C) \cup (B \cap C)$

<u>Case 1:</u> $x \in A \cap C$

$\Rightarrow x \in A$ and $x \in C$.

$\Rightarrow x \in A \cup B$ and $x \in C$.

$\Rightarrow x \in (A \cup B) \cap C$.

<u>Case 2:</u> $x \in B \cap C$

$\Rightarrow x \in B$ and $x \in C$.

$\Rightarrow x \in A \cup B$ and $x \in C$.

$\Rightarrow x \in (A \cup B) \cap C$.

In both cases, $x \in (A \cup B) \cap C$ so <u>we are done.</u>

<u>A.2.1</u> d) Show $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

First show $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$

$x \in (A \cap B) \cup C$

Case 1: $x \in A \cap B$.

$\Rightarrow x \in A$ and $x \in B$.

$\Rightarrow x \in A \cup C$ and $x \in B \cup C$

$\Rightarrow \quad x \in (A \cup C) \cap (B \cup C)$

Case 2: $\quad x \in C$

$\Rightarrow x \in A \cup C$ and $x \in B \cup C$.

$\Rightarrow \quad x \in (A \cup C) \cap (B \cup C)$.

In both cases, $x \in (A \cup C) \cap (B \cup C)$.

Next, show $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$ in a similar way.

A.2.1 e) Show $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

First show $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$.

Consider $x \in (A \cup B) \setminus C$.

$\therefore \quad x \in A \cup B$ and $x \notin C$.

Case 1: $\quad x \in A$

$\Rightarrow x \in A$ and $x \notin C$.

$\Rightarrow \quad x \in A \setminus C$

$\Rightarrow \quad x \in (A \setminus C) \cup (B \setminus C)$.

Case 2: $\quad x \in B$

$\Rightarrow x \in B$ and $x \notin C$.

$\Rightarrow x \in B \setminus C$

$\Rightarrow x \in (B \setminus C) \cup (A \setminus C)$

In both cases, $x \in (A \setminus C) \cup (B \setminus C)$.

Next show $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$

$x \in (A \setminus C) \cup (B \setminus C)$

Case 1 : $x \in A \setminus C$

$\Rightarrow x \in A$ and $x \notin C$.

$\Rightarrow x \in A \cup B$ and $x \notin C$

$\Rightarrow x \in (A \cup B) \setminus C$.

Case 2 : $x \in B \setminus C$

$\Rightarrow x \in B$ + $x \notin C$

$\Rightarrow x \in A \cup B$ + $x \notin C$

$\Rightarrow x \in (A \cup B) \setminus C$.

In both cases, $x \in (A \cup B) \setminus C$ Done.

A.2.1 f) Similar to A.2.1.e).

A.2.1 g) Show $\{x \in \mathbb{R} : x^2 + x < 0\} = (-1, 0)$.

First show $\{x \in \mathbb{R} : x^2 + x < 0\} \subseteq (-1, 0)$.

Need to show $x \in \mathbb{R}$ such that $x^2 + x < 0$

$\Rightarrow x \in (-1, 0)$.

Show contrapositive:

$x \notin (-1, 0) \Rightarrow x \notin \{x \in \mathbb{R} : x^2 + x < 0\}$.

Case 1: $x \leq -1$

$\Rightarrow$ ~~x(x+1)~~ $x \leq 0$ and $x + 1 \leq 0$

$\Rightarrow x(x+1) \geq 0$ (product of 2 nonpositive numbers is nonnegative)

$\Rightarrow x^2 + x \geq 0$

$\Rightarrow x \notin \{x \in \mathbb{R} : x^2 + x < 0\}$

Case 2: $x \geq 0$.

$\Rightarrow \quad x \geq 0$ and $x+1 \geq 0$

$\Rightarrow \quad x(x+1) \geq 0$ (product of 2 nonnegative numbers is nonnegative)

$\Rightarrow \quad x \notin \{x \in \mathbb{R} : x^2+x<0\}$.

In both cases, $x \notin \{x \in \mathbb{R} : x^2+x<0\}$.

Thus, $x \notin (-1,0) \Rightarrow x \notin \{x \in \mathbb{R} : x^2+x<0\}$.

$\therefore$, $\{x \in \mathbb{R} : x^2+x<0\} \subseteq (-1,0)$.

Next, show $(-1,0) \subseteq \{x \in \mathbb{R} : x^2+x<0\}$.

$x \in (-1,0)$

$\Rightarrow \quad -1 < x$ and $x < 0$.

$\Rightarrow \quad 0 < x+1$ and $x < 0$.

$\Rightarrow \quad x(x+1) < 0$ (product of negative and positive is negative).

$\Rightarrow \quad x \in \{x \in \mathbb{R} : x^2+x<0\}$.

Done.

A.2.2. a) $\bigcup_{n=1}^{N}\left(-\frac{1}{n}, \frac{1}{n}\right) = \left(-1, 1\right)$

and

$\bigcap_{n=1}^{N}\left(-\frac{1}{n}, \frac{1}{n}\right) = \left(-\frac{1}{N}, \frac{1}{N}\right)$

b) $\bigcup_{n=1}^{N}(-n, n) = (-N, N)$

$\bigcap_{n=1}^{N}(-n, n) = (-1, 1)$

c) $\bigcup_{n=1}^{N}[n, n+1] = [1, N].$

$$\bigcap_{n=1}^{N}[n, n+1] = [1, 2] \quad \text{if } N = 1.$$
$$= \{2\} \quad \text{if } N = 2.$$
$$= \emptyset \quad \text{if } N \geq 3.$$

A.2.3. a) $\bigcup_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = (-1, 1)$

Pf: Show first $\bigcup_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq (-1, 1)$

Consider $x \in \bigcup_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$

$\Rightarrow x \in \left(-\frac{1}{i}, \frac{1}{i}\right)$ for some $i \in \mathbb{N}$

$$\Rightarrow \quad -\frac{1}{i} < x < \frac{1}{i}$$

$$\Rightarrow \quad -1 \leq -\frac{1}{i} < x < \frac{1}{i} \leq 1$$

$$\Rightarrow \quad x \in (-1,1).$$

Next show $(-1,1) \subseteq \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$

Consider $x \in (-1,1)$

$$\Rightarrow x \in (-1,1) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{1}{3}, \frac{1}{3}\right) \cup \ldots$$

$$\Rightarrow x \in \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) \qquad \text{Done.}$$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

Pf: Show first $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq \{0\}$.

Need to show: $x \in \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) \Rightarrow x = 0$

Show contrapositive:

$$x \neq 0 \Rightarrow x \notin \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right).$$

Case 1: $x > 0$. Then there exists $i \in \mathbb{N}$ s.t. $\frac{1}{i} \leq x$.

$$\Rightarrow x \notin \left(-\frac{1}{i}, \frac{1}{i}\right) \Rightarrow x \notin \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

Case 2: $x<0$. Then there exists $i \in \mathbb{N}$ s.t.

$$x \leq -\frac{1}{i}$$

$$\Rightarrow x \notin \left(-\frac{1}{i}, \frac{1}{i}\right) \Rightarrow x \notin \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$$

In both cases $x \notin \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$. Done ✓

Next, show $\{0\} \subseteq \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$.

Consider $x=0$.

$$\Rightarrow -\frac{1}{n} < x < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

$$\Rightarrow x \in \left(-\frac{1}{n}, \frac{1}{n}\right) \quad \forall\, n \in \mathbb{N}.$$

$$\Rightarrow x \in \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right).$$ Done.

A.2.3 b) $\bigcup_{n=1}^{\infty}(-n,n) = \mathbb{R}\setminus\mathbb{Z}$

= real numbers without the integers.

$$\bigcap_{n=1}^{\infty}(-n,n) = \phi.$$

A.2.3 c) $\bigcup_{n=1}^{\infty}[n,n+1] = \{x \in \mathbb{R} : x \geq 1\}.$

$$\bigcap_{n=1}^{\infty}[n,n+1] = \phi$$

2. De Morgan's laws:

i) $A \setminus \left(\bigcup_{i \in I} B_i \right) = \bigcap_{i \in I} (A \setminus B_i)$

First show $A \setminus \bigcup_{i \in I} B_i \subseteq \bigcap_{i \in I} (A \setminus B_i)$

$x \in A \setminus \bigcup_{i \in I} B_i$

$\Rightarrow x \in A \;\; + \;\; x \notin \bigcup_{i \in I} B_i$

$\Rightarrow x \in A \;\; + \;\; x \notin B_i$ for any $i \in I$.

$\Rightarrow x \in A \setminus B_i \quad \forall i \in I$.

$\Rightarrow x \in \bigcap_{i \in I} (A \setminus B_i)$

Next show $\bigcap_{i \in I} (A \setminus B_i) \subseteq A \setminus \left(\bigcup_{i \in I} B_i \right)$

$x \in \bigcap_{i \in I} A \setminus B_i$

$\Rightarrow x \in A \setminus B_i \quad \forall i \in I$

$\Rightarrow x \in A$ and $x \notin B_i \quad \forall \in I$.

$\Rightarrow x \in A$ and $x \notin \bigcup_{i \in I} B_i$

$\Rightarrow x \in A \setminus \left(\bigcup_{i \in I} B_i \right)$

2 ii) $$A \setminus \left(\bigcap_{i \in I} B_i \right) = \bigcup_{i \in I} (A \setminus B_i)$$

First show $A \setminus \left(\bigcap_{i \in I} B_i \right) \subseteq \bigcup_{i \in I} (A \setminus B_i)$

$x \in A \setminus \left(\bigcap_{i \in I} B_i \right)$

$\Rightarrow x \in A$ and $x \notin \bigcap_{i \in I} B_i$

$\Rightarrow x \in A$ and $x \notin B_i$ for at least some $i \in I$.

$\Rightarrow x \in A \setminus B_i$ for some $i \in I$.

$\Rightarrow x \in \bigcup_{i \in I} (A \setminus B_i)$

Next show $\bigcup_{i \in I} (A \setminus B_i) \subseteq A \setminus \left(\bigcap_{i \in I} B_i \right)$

$x \in \bigcup_{i \in I} A \setminus B_i$

$\Rightarrow x \in A \setminus B_i$ for some $i \in I$.

$\Rightarrow x \in A$ and $x \notin B_i$ for some $i \in I$.

$\Rightarrow x \in A$ and $x \notin \bigcap_{i \in I} B_i$

$\Rightarrow x \in A \setminus \left(\bigcap_{i \in I} B_i \right)$

$\blacksquare$.

3. First show ~~$A \cup B$~~

<u>Claim 1</u>: $A \cup B = A \cup (B \setminus A)$

<u>Pf</u>: Show $A \cup B \subseteq A \cup (B \setminus A)$.

$x \in A \cup B$

$\Rightarrow x \in A$ or $x \in B$.

<u>Case 1</u>: $x \in A$

$\Rightarrow x \in A \cup (B \setminus A)$

<u>Case 2</u>: $x \in B$

If $x \in A$ also, then by Case 1 we are done.

$\therefore$ we assume $x \notin A$.

$\Rightarrow x \in B$ and $x \notin A$.

$\Rightarrow x \in B \setminus A$.

$\Rightarrow x \in A \cup (B \setminus A)$. $\blacksquare$

<u>Claim 2</u>: $A \cap (B \setminus A) = \phi$.

Consider $x \in A$ and show $x \notin B \setminus A$.

By contradiction: Assume $x \in B \setminus A$.

$\Rightarrow x \in B$ and $x \notin A$.

But this contradicts $x \in A$. $\blacksquare$

$\therefore$ By <u>claim 1</u> and <u>claim 2</u> and the cardinality property of disjoint sets, we have

$$|A \cup B| = |A \cup (B \setminus A)| = |A| + |B \setminus A|. \quad -(1).$$

Claim 3: $(B \setminus A) \cup (A \cap B) = B$.

Pf: First show $(B \setminus A) \cup (A \cap B) \subseteq B$.

$x \in (B \setminus A) \cup (A \cap B)$

$\Rightarrow x \in B \setminus A$ or $x \in A \cap B$.

Case 1: $x \in B \setminus A$

$\Rightarrow x \in B$ and $x \notin A$

$\Rightarrow x \in B$.

Case 2: $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$

In both cases we have $x \in B$.

Next show $B \subseteq (B \setminus A) \cup (A \cap B)$

Consider $x \in B$

Consider 2 exhaustive cases:

Case 1: $x \in A$

$\Rightarrow x \in A$ and $x \in B$.

$\Rightarrow x \in A \cap B$

$\Rightarrow x \in (A \cap B) \cup (B \setminus A)$

Case 2: $x \notin A$.

$\Rightarrow x \in B$ and $x \notin A$

$\Rightarrow x \in B \setminus A$.

$\Rightarrow x \in (A \cap B) \cup (B \setminus A)$.

In both cases, $x \in (A \cap B) \cup (B \setminus A)$.

Claim 4: $(A \cap B) \cap (B \setminus A) = \emptyset$.

Pf: Consider $x \in B \setminus A$ and show $x \notin A \cap B$.

$x \in B \setminus A$

$\Rightarrow x \in B$ and $x \notin A$.

$\Rightarrow x \notin A \cap B$. Done.

By claim 3 + Claim 4 and disjoint set property,

$$|(B \setminus A) \cup (A \cap B)| = |B|$$

$$\Rightarrow |B \setminus A| + |A \cap B| = |B|.$$

$$\Rightarrow |B \setminus A| = |B| - |A \cap B|.$$

Using equation (1):

$$|A \cup B| = |A| + |B \setminus A| = |A| + |B| - |A \cap B|$$

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4. Show $|(0,1)| = |(a,b)|$ for $a \neq b$.

Need to show $|(0,1)| \leq |(a,b)|$ and $|(a,b)| \leq |(0,1)|$

First show $|(0,1)| \leq |(a,b)|$.

Define $f : (0,1) \to (a,b)$

by $f(x) = a + (b-a)x$.

Show f is injective: $f(x) = f(y)$

$\Rightarrow a + (b-a)x = a + (b-a)y$

$\Rightarrow (b-a)x = (b-a)y$

$\Rightarrow x = y$ (since $b-a \neq 0$)

Also show that $\forall x \in (0,1)$, $f(x) \in (a,b)$.

$$x \in (0,1)$$

$$\Rightarrow \quad 0 < x < 1$$

$$\Rightarrow \quad 0 < (b-a)x < b-a$$

$$\Rightarrow \quad a < a + (b-a)x < a + (b-a)$$

$$\Rightarrow \quad a < a+(b-a)x < b$$

$$\Rightarrow \quad a+(b-a)x \in (a,b)$$

$$\Rightarrow \quad f(x) \in (a,b).$$

<u>Next show $|(a,b)| \leq |(0,1)|$</u>

Define $g : (a,b) \to (0,1)$ by

$$g(x) = \frac{x-a}{b-a}$$ [this is well-defined since $b-a \neq 0$.].

Show g is injective:

$$g(x) = g(y) \Rightarrow \frac{x-a}{b-a} = \frac{y-a}{b-a}$$

$$\Rightarrow x-a = y-a \quad (\text{since } b-a \neq 0)$$

$$\Rightarrow x = y$$

Also show $\forall x \in (a,b)$, $g(x) \in (0,1)$:

$$a < x < b$$

$$\Rightarrow 0 < x-a < b-a$$

$$\Rightarrow 0 < \frac{x-a}{b-a} < \frac{b-a}{b-a}$$

$$\Rightarrow 0 < g(x) < 1 \quad \Rightarrow \quad g(x) \in (0,1).$$

~~Cla~~ Claim: $|(0,1)| = |[a,b]|$

First $|(0,1)| = |(a,b)| \leq |[a,b]|$

since $(a,b) \subseteq [a,b]$ and the inequality follows from Exercise 5.

Next show $|[a,b]| \leq |(0,1)|$

Define $f : [a,b] \to (0,1)$.

by $f(x) = \dfrac{2x+b-3a}{4b-4a}$ [Many different functions work]

First show $\forall x \in [a,b]$, $f(x) \in (0,1)$:

$a \leq x \leq b$

$\Rightarrow 0 \leq x-a \leq b-a.$

$\Rightarrow 0 \leq 2(x-a) \leq 2(b-a)$

Now since $b-a > 0$ and $2(x-a) \geq 0$,

$0 < 2(x-a)+(b-a) \leq 2(b-a)+(b-a)$

$\Rightarrow 0 < 2(x-a)+(b-a) \leq 3(b-a) < 4(b-a)$

$\Rightarrow 0 < 2x+b-3a < 4(b-a)$

$\Rightarrow 0 < \dfrac{2x+b-3a}{4b-4a} < 1.$

Also, we can show $f(x)$ is injective:

$f(x) = f(y) \Rightarrow \dfrac{2x+b-3a}{4b-4a} = \dfrac{2y+b-3a}{4b-4a} \Rightarrow 2x+\cancel{b-3a} = 2y+\cancel{b-3a}$
$\Rightarrow 2x = 2y \Rightarrow \underline{x=y}$

5. Show $A \subseteq B \Rightarrow |A| \leq |B|$

Pf: Define $f(x) = x$ as the injective map

$$f : A \to B.$$

First, show $f(x) \in B \quad \forall x \in A.$

since $f(x) = x \in A$

$\Rightarrow x \in B$ since $A \subseteq B$.

Next show $f(x)$ is injective:

$$f(x) = f(y)$$

$$\Rightarrow x = y \qquad [\text{by definition } f(x) = x].$$

6. Show $|A| \leq |B| \leq |C| \Rightarrow |A| \leq |C|$.

Since $|A| \leq |B|$

$\Rightarrow \exists$ injective $f : A \to B$.

Since $|B| \leq |C|$

$\exists$ injective $g : B \to C$.

Consider $h = g \circ f$. Of course, $h : A \to C$.

Show h is injective:

$h(x) = h(y)$

$\Rightarrow g(f(x)) = g(f(y))$

$\Rightarrow f(x) = f(y)$ since g is injective.

$\Rightarrow x = y$ since f is injective.

Done.

7 (i). I is countable.

$\Rightarrow \exists$ injective $f : I \longrightarrow \mathbb{N}$.

For every $j \in \mathbb{N} = \{1, 2, 3, \dots$

If $\exists\, i \in I$ such that $f(i) = j$

Define $C_j = A_i$

Else define $C_j = \phi$.

Proof that C_j is countable is immediate

Need to show

$$\bigcup_{j \in \mathbb{N}} C_j = \bigcup_{i \in I} A_i$$

Show $\bigcup_{j \in \mathbb{N}} C_j \subseteq \bigcup_{i \in I} A_i$

$x \in \bigcup_{j \in \mathbb{N}} C_j \Rightarrow x \in C_j$ for some $j \in \mathbb{N}$.

$\Rightarrow C_j \neq \phi$

$\Rightarrow \exists\, i \in I$ such that $f(i) = j$

$\Rightarrow C_j = A_i$

$\Rightarrow x \in A_i$

$\Rightarrow x \in \bigcup_{i \in I} A_i$

Next show $\bigcup_{i \in I} A_i \subseteq \bigcup_{j \in \mathbb{N}} C_j$

$x \in \bigcup_{i \in I} A_i \Rightarrow x \in A_i$ for some $i \in I \Rightarrow x \in C_j$ where $j = f(i)$.

$\Rightarrow x \in \bigcup_{j \in \mathbb{N}} C_j$

7 (ii) Construct B_j $j = 1, 2, \ldots$

s.t. 1. $B_i \cap B_j = \phi$ when $i \neq j$.

2. B_j is countable for all $j \in \mathbb{N}$.

3. $\bigcup_{j \in \mathbb{N}} B_j = \bigcup_{j \in \mathbb{N}} C_j$

Define $B_1 = C_1$

and $B_j = C_j \setminus (C_1 \cup C_2 \cup \ldots C_{j-1})$

for $j \geq 2$.

<u>Show 1. holds</u>: $B_i \cap B_j = \phi$ when $i \neq j$.

Let $i < j$.

Show $x \in B_j \Rightarrow x \notin B_i$

$x \in B_j \Rightarrow x \in C_j \setminus (C_1 \cup \ldots C_{j-1})$

$\Rightarrow x \notin C_i$ (since $i < j$).

$\Rightarrow x \notin C_i \setminus (C_1 \ldots C_{i-1})$

$\Rightarrow x \notin B_i$

<u>Show 2. holds</u>: Since $B_j \subseteq C_j$ by exercise 5, $|B_j| \leq |C_j|$

and $|C_j| \leq |\mathbb{N}|$ by assumption.

By exercise 6, $|B_j| \leq |\mathbb{N}|$.

<u>Show 3. holds</u>:

Show $\bigcup_{j\in\mathbb{N}} B_j \subseteq \bigcup_{j\in\mathbb{N}} C_j$

$x \in \bigcup_{j\in\mathbb{N}} B_j \Rightarrow x \in B_j$ for some $j \in \mathbb{N}$.

$\Rightarrow x \in C_j \setminus (C_1 \cup \dots C_{j-1})$ by def. of B_j

$\Rightarrow x \in C_j$

$\Rightarrow x \in \bigcup_{j\in\mathbb{N}} C_j$.

Next show $\bigcup_{j\in\mathbb{N}} C_j \subseteq \bigcup_{j\in\mathbb{N}} B_j$

Consider $x \in \bigcup_{j\in\mathbb{N}} C_j$

$\Rightarrow x \in C_j$ for at least some $j \in \mathbb{N}$.

Let j^* be the minimum $j \in \mathbb{N}$ s.t. $x \in C_j$

$\therefore$ $x \in C_{j^*}$ and $x \notin C_1$, $x \notin C_2 \dots x \notin C_{j^*-1}$

$\Rightarrow x \in C_{j^*} \setminus (C_1 \cup C_2 \cup \dots C_{j^*-1})$

$\Rightarrow x \in B_{j^*}$ by def. of B_{j^*}.

$\Rightarrow x \in \bigcup_{j\in\mathbb{N}} B_j$

$\blacksquare$

7 iii) Since each B_i is countable,

there is an injective map $g_i : B_i \to \mathbb{N}$.

First construct an injective map.

$$\tilde{f} : \bigcup_{i \in \mathbb{N}} B_i \longrightarrow \mathbb{N} \times \mathbb{N}$$

defined as follows.

For any $x \in \bigcup_{i \in \mathbb{N}} B_i$, there exists a

unique $i \in \mathbb{N}$ s.t. $x \in B_i$

since $B_i \cap B_j = \phi$ when $i \neq j$.

Define $\tilde{f}(x) = (i, g_i(x))$.

Show $\tilde{f}$ is injective:

$\tilde{f}(x) = \tilde{f}(y)$

$\Rightarrow (i_1, g_{i_1}(x)) = (i_2, g_{i_2}(y))$

$\Rightarrow i_1 = i_2$ and $g_{i_1}(x) = g_{i_2}(y)$.

But g_{i_1} is an injective map $g_{i_1} = g_{i_2}$ since $i_1 = i_2$.

$\Rightarrow x = y$.

Now we know there is an injective map

$h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ since $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$

Define $f : \bigcup_{i \in \mathbb{N}} B_i \longrightarrow \mathbb{N}$ by $f = h \circ \tilde{f}$

Since h and $\tilde{f}$ are injective,

f is injective.

$$\therefore \left|\bigcup_{i\in\mathbb{N}} B_i\right| \leq |\mathbb{N}|$$

$$\Rightarrow \bigcup_{i\in\mathbb{N}} B_i \text{ is countable.}$$

7(iv) We showed $\bigcup_{i\in I} A_i = \bigcup_{j\in\mathbb{N}} B_j$ (parts (i) + (ii))

and $\left|\bigcup_{j\in\mathbb{N}} B_j\right| \leq |\mathbb{N}|$ by part (iii)

<u>Claim</u>: $A = B \Rightarrow |A| = |B|$ for any sets A, B.

Pf: $A = B \Rightarrow A \subseteq B \Rightarrow |A| \leq |B|$ by Exercise 5.

Also $A = B \Rightarrow B \subseteq A \Rightarrow |B| \leq |A|$ " " "

$\Rightarrow |A| = |B|$.

$$\therefore \text{ since } \bigcup_{i\in I} A_i = \bigcup_{j\in\mathbb{N}} B_j$$

$$\Rightarrow \left|\bigcup_{i\in I} A_i\right| = \left|\bigcup_{j\in\mathbb{N}} B_j\right| \text{ by the Claim above.}$$

Since $\left|\bigcup_{j\in\mathbb{N}} B_j\right| \leq |\mathbb{N}|$

we have $\left|\bigcup_{i\in I} A_i\right| \leq |\mathbb{N}|$ by Exercise 6.

8. The set of all functions $f : \mathbb{N} \to \{0,1\}$ is <u>NOT</u> countable. If you want to see why come <u>talk to me</u>.

9. Define
$$A_1 = \{x \in \mathbb{Q} : x > 0\}.$$
$$A_2 = \{x \in \mathbb{Q} : x < 0\}.$$
$$A_3 = \{0\}.$$

Clearly, $A_1 \cup A_2 \cup A_3 = \mathbb{Q}$ [show it!].

Thus $|A_1 \cup A_2 \cup A_3| = |\mathbb{Q}|$ by the Claim in 7 (iv).

~~The first the~~ <u>Claim 1</u>: $|A_1| \leq |\mathbb{N}|$.

We show an injective map $f : A_1 \to \mathbb{N} \times \mathbb{N}$.

for every $x \in A_1$ since $x \in \mathbb{Q}$ and $x > 0$, there exist natural numbers m, n such $x = \frac{m}{n}$

Moreover consider the lowest form, i.e., m and n have no common factors.

Define $f(x) = (m, n)$.

Easy to see it is injective: $f(x) = f(y)$
$$\Rightarrow (m_1, n_1) = (m_2, n_2)$$
$$\Rightarrow m_1 = m_2 \text{ and } n_1 = n_2$$
$$\Rightarrow m_1/n_1 = m_2/n_2 \Rightarrow x = y.$$

Thus, $|A_1| \leq |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

$\therefore |A_1| \leq |\mathbb{N}|$ by Exercise 6. $\boxtimes$

<u>Claim 2</u>: $|A_2| \leq |\mathbb{N}|$

Define $g : A_2 \longrightarrow A_1$

by $g(x) = -x$.

First show $g(x) \in A_1 \quad \forall\, x \in A_2$.

$x \in A_2 \Rightarrow x < 0$ and $x \in \mathbb{Q}$

$\Rightarrow -x > 0$ and $-x \in \mathbb{Q}$

$\Rightarrow -x \in A_1$

$\Rightarrow g(x) \in A_1$.

Clearly injective [Check!]. $\boxtimes$

$\therefore |A_2| \leq |A_1| \leq |\mathbb{N}|$ Using Claim 1 + 2.

$\Rightarrow |A_2| \leq |\mathbb{N}|$ by Exercise 6. $\boxtimes$

<u>Claim 3</u>: $|A_3| \leq |\mathbb{N}|$ Pf: A_3 is a finite set.

$\therefore$ By Exercise 7,

$A_1 \cup A_2 \cup A_3$ is countable since each A_i is countable by Claims 1, 2 + 3.

and $I = \{1, 2, 3\}$ is a finite set + therefore countable.

10. Suppose to the contrary that $|\mathbb{Q}| = |\mathbb{R}|$. Thus, $|\mathbb{R}| \leq |\mathbb{Q}|$ and $|\mathbb{Q}| \leq |\mathbb{R}|$

Since $[0,1] \subseteq \mathbb{R}$.

$$|[0,1]| \leq |\mathbb{R}| \quad \text{by Exercise 5.}$$

$$\therefore \quad |[0,1]| \leq |\mathbb{R}| \leq |\mathbb{Q}| \leq |\mathbb{N}|$$

where the last inequality follows from Exercise 9.

$$\therefore \quad |[0,1]| \leq |\mathbb{N}|$$

This is a contradiction because we showed in class that $|[0,1]| \nleq |\mathbb{N}|$

$\therefore \quad |\mathbb{Q}| \neq |\mathbb{R}|$.

11. The element 3 does not have a multiplicative inverse, i.e., $\nexists\, x \in X$ s.t.

$$(3)(x) = 1.$$

This contradicts field axiom 8 (M4 in text).

[Note that the element 4 also has the same problem].

12 i) $(a+b)^2 = (a+b)(a+b)$

$= (a+b)a + (a+b)b$ [Axiom 9].

$= a\cdot a + ba + a\cdot b + b\cdot b$ [Axiom 9].

$= a^2 + ab + ab + b^2$ [Axiom 5].

ii) $(a+(-a))\cdot b = 0\cdot b$ [~~Axiom~~ 3].

$= 0$ [Consequence 1].

Also $(a+(-a))\cdot b = (a\cdot b) + (-a)\cdot(b)$ [Axiom 9].

$\therefore\ 0 = (a\cdot b) + (-a)(b)$.

$\therefore\ -(a\cdot b) = (-a)(b)$ by definition of additive inverse or Axiom 3

iii) $(a' + a) + a'' \underset{\text{given}}{=\!=} 0 + a'' \underset{\text{axiom 3}}{=\!=} a''$.

Also $(a'+a)+a'' \underset{\text{Axiom 2}}{=\!=} a' + (a+a'') \underset{\text{given}}{=\!=} a'+0$

$\underset{\text{axiom 3}}{=\!=} a'$

$\therefore\ a'' = (a'+a)+a'' = a'$

$\Rightarrow a'' = a'$.

Since $\quad a + (-a) = (-a) + a = 0.$

~~a is the additive inverse $-(-a)$~~

a satisfies the property for $-(-a)$

Since there is a **unique** additive inverse for $-a$, a __is__ this unique inverse.

iv) $$(-1)(-1)$$

$$= -(1)(-1) \qquad \text{[Exercise 12(ii)]}$$

$$= -(-1)$$

$$= 1 \qquad \text{[Exercise 12 (iii)]}.$$