Homework Problems : WEEK X

For any sequence $\{x_n\}_{n=1}^{\infty}$, we say $\beta \in \mathbb{R}$ is an *eventual upper bound* of $\{x_n\}_{n=1}^{\infty}$ if there exists $N \in \mathbb{N}$ such that $x_n < \beta$ for all $n \geq N$. Similarly, we say $\alpha \in \mathbb{R}$ is an *eventual lower bound* of $\{x_n\}_{n=1}^{\infty}$ if there exists $N \in \mathbb{N}$ such that $\alpha < x_n$ for all $n \geq N$.

1. Show that if $\sum_{k=1}^{\infty} a_k$ converges, $\lim_{k\to\infty} a_k \to 0$. [Hint: Use the Cauchy criterion for convergence of the partial sums s_n and the fact that $a_n = s_n - s_{n-1}$ for all $n \geq 2$]

2. *This is a problem about sequences that is useful for the next problem.* Let $\{s_n\}_{n=1}^{\infty}$ be any sequence. Let N be a fixed natural number. Define a new sequence $s'_n = s_{n+N}$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} s'_n = \lim_{n\to\infty} s_n$. (This is sometimes rephrased as "A sequence converges if and only if its *tail* converges")

3. Let $\sum_{k=1}^{\infty} a_k$ be a series. Let N be a fixed nonnegative integer (i.e., $N \geq 0$). Define a sequence $b_k = a_{N+k}$ for all $k \in \mathbb{N}$. Let $s_n = \sum_{k=1}^{n} a_k$ be the partial sums for the series $\sum_{k=1}^{\infty} a_k$, and let $s'_n = \sum_{k=1}^{n} b_k$ be the partial sums for the series $\sum_{k=1}^{\infty} b_k$.

   (i) Show that $\exists A \in \mathbb{R}, \forall n \in \mathbb{N}, s_{n+N} = A + s'_n$.

   (ii) Show that $\{s_n\}_{n=1}^{\infty}$ converges if and only if $\{s'_n\}_{n=1}^{\infty}$ converges, and $\lim_{n\to\infty} s_n = A + \lim_{n\to\infty} s'_n$.

   (iii) Show that $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges. What is the relationship between these two infinite sums ? (This is sometimes rephrased as "A series converges if and only if its *tail* converges")

4. Do 3.5.1. [Hint: Problems 1 and 3 might be useful]

   Suppose $\sum_{k=1}^{\infty} a_k$ converges. We say that $\sum_{k=1}^{\infty} a_k$ converges *absolutely* if $\sum_{k=1}^{\infty} |a_k|$ converges. We say $\sum_{k=1}^{\infty} a_k$ converges *nonabsolutely* if $\sum_{k=1}^{\infty} |a_k|$ diverges. Thus Dirichlet's theorem can be restated as :

   Dirichlet's Theorem : If a series $\sum_{k=1}^{\infty} a_k$ converges aboslutely, thenn $\sum_{k=1}^{\infty} a_k$ converges unconditionally.

5. Do 3.7.9 (Show that the sequence converges unconditionally) [Hint: Think about if the series converges absolutely and use Dirichlet's theorem above], 3.7.10 [Hint: Use Dirichlet's theorem].