For any sequence \( \{x_n\}_{n=1}^{\infty} \), we say \( \beta \in \mathbb{R} \) is an eventual upper bound of \( \{x_n\}_{n=1}^{\infty} \) if there exists \( N \in \mathbb{N} \) such that \( x_n < \beta \) for all \( n \geq N \). Similarly, we say \( \alpha \in \mathbb{R} \) is an eventual lower bound of \( \{x_n\}_{n=1}^{\infty} \) if there exists \( N \in \mathbb{N} \) such that \( \alpha < x_n \) for all \( n \geq N \).

1. Show that if \( \sum_{k=1}^{\infty} a_k \) converges, \( \lim_{k \to \infty} a_k \to 0 \). [Hint: Use the Cauchy criterion for convergence of the partial sums \( s_n \) and the fact that \( a_n = s_n - s_{n-1} \) for all \( n \geq 2 \)]

2. This is a problem about sequences that is useful for the next problem. Let \( \{s_n\}_{n=1}^{\infty} \) be any sequence. Let \( N \) be a fixed natural number. Define a new sequence \( s'_n = s_{n+N} \) for all \( n \in \mathbb{N} \). Show that \( \lim_{n \to \infty} s'_n = \lim_{n \to \infty} s_n \). (This is sometimes rephrased as “A sequence converges if and only if its tail converges”)

3. Let \( \sum_{k=1}^{\infty} a_k \) be a series. Let \( N \) be a fixed nonnegative integer (i.e., \( N \geq 0 \)). Define a sequence \( b_k = a_{N+k} \) for all \( k \in \mathbb{N} \). Let \( s_n = \sum_{k=1}^{n} a_k \) be the partial sums for the series \( \sum_{k=1}^{\infty} a_k \), and let \( s'_n = \sum_{k=1}^{n} b_k \) be the partial sums for the series \( \sum_{k=1}^{\infty} b_k \).

   (i) Show that \( \exists A \in \mathbb{R}, \forall n \in \mathbb{N}, s_{n+N} = A + s'_n \).

   (ii) Show that \( \{s_n\}_{n=1}^{\infty} \) converges if and only if \( \{s'_n\}_{n=1}^{\infty} \) converges, and \( \lim_{n \to \infty} s_n = A + \lim_{n \to \infty} s'_n \).

   (iii) Show that \( \sum_{k=1}^{\infty} a_k \) converges if and only if \( \sum_{k=1}^{\infty} b_k \) converges. What is the relationship between these two infinite sums? (This is sometimes rephrased as “A series converges if and only if its tail converges”)

4. Do 3.5.1. [Hint: Problems 1 and 3 might be useful]

Suppose \( \sum_{k=1}^{\infty} a_k \) converges. We say that \( \sum_{k=1}^{\infty} a_k \) converges absolutely if \( \sum_{k=1}^{\infty} |a_k| \) converges. We say \( \sum_{k=1}^{\infty} a_k \) converges nonabsolutely if \( \sum_{k=1}^{\infty} |a_k| \) diverges. Thus Dirichlet’s theorem can be restated as :

Dirichlet’s Theorem : If a series \( \sum_{k=1}^{\infty} a_k \) converges absolutely, then \( \sum_{k=1}^{\infty} a_k \) converges unconditionally.

5. Do 3.7.9 (Show that the sequence converges unconditionally) [Hint: Think about if the series converges absolutely and use Dirichlet’s theorem above], 3.7.10 [Hint: Use Dirichlet’s theorem].