HW 10 Solutions MAT 25.

1. Since $\sum_{k=1}^{\infty} a_k$ converges, the sequence of partial sums $S_n = \sum_{k=1}^{n} a_k$ converges.

$\Rightarrow \{S_n\}$ is a Cauchy sequence by the Cauchy Criterion theorem.

$\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|S_n - S_m| < \epsilon \ \forall n, m \geq N$. (*)

Want to show: $a_k \to 0$

So need to show: $\forall \gamma > 0, \exists N \in \mathbb{N}, |a_n - 0| < \gamma \ \forall n \geq N$

1. Consider $\gamma > 0$.
   Using $\epsilon = \gamma$ on (*), we get $N_1 \in \mathbb{N}$ s.t. $|S_n - S_m| < \epsilon \ \forall n, m \geq N_1$

2. Set $N = N_1 + 1$
3. Consider $n \geq N$.

$$
\begin{aligned}
|a_n - 0| &= |a_n| \\
&= |S_n - S_{n-1}| \\
&< \epsilon \quad \text{since } n \geq N \geq N_1 \text{ and } n-1 \geq N-1 \geq N_1 \\
&= \gamma
\end{aligned}
$$

$\Rightarrow |a_n - 0| < \gamma \quad \forall n \geq N$. ∎

Assume $\{s_n\}_{n=1}^{\infty}$ converges.

2. Let $\lim_{n\to\infty} s_n = L$.

So we know: $\forall \epsilon > 0, \exists N_1 \in \mathbb{N}$ s.t. $|s_n - L| < \epsilon \ \forall n \geq N_1$

Want to show: $\lim_{n\to\infty} s_{n+N} = L$.

Need to show: $\forall \gamma > 0, \exists N_2 \in \mathbb{N}$ s.t. $|s_{n+N} - L| < \gamma$
$\forall n \geq N_2$.

1. Consider $\gamma > 0$.
2. Using $\epsilon = \gamma$ on def. of $\lim_{n\to\infty} s_n = L$,

   get $N_1 \in \mathbb{N}$ s.t. $|s_n - L| < \gamma \ \forall n \geq N_1$. (*)

3. Set $N_2 = N_1$.
4. Choose $n \geq N_2$.
   $\Rightarrow |s_{n+N} - L| < \gamma$ (since $n+N \geq n \geq N_2 = N_1$) using (*).

$\blacksquare$

Assume $\{s_{n+N}\}_{n=1}^{\infty}$ converges.

Let $\lim_{n\to\infty} s_{n+N} = L$.

So we know: $\forall \epsilon > 0, \exists N_1 \in \mathbb{N}$ s.t. $|s_{n+N} - L| < \epsilon \ \forall n \geq N_1$.

Need to show: $\forall \gamma > 0, \exists N_2 \in \mathbb{N}$ s.t. $|s_n - L| < \gamma \ \forall n \geq N_2$.

1. Consider $\gamma > 0$.
2. Using $\epsilon = \gamma$ on def. of $\lim_{n\to\infty} s_{n+N} = L$

get $N_1 \in \mathbb{N}$ s.t. $|s_{n+N} - L| < \gamma \quad \forall n \geq N_1$. (*)

3. Set $N_2 = N_1 + N$.
4. Consider $n \geq N_2$.

$$\Rightarrow n \geq N_1 + N$$

$$\Rightarrow n - N \geq N_1$$

$$\Rightarrow |s_{(n-N)+N} - L| < \gamma \quad (\text{using } (*))$$

$$\Rightarrow |s_n - L| < \gamma. \quad \blacksquare$$

Thus, $\{s_n\}$ converges $\Leftrightarrow$ $\{s_{n+N}\}$ converges.
and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+N}$.

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3. (i) Let $A = \sum_{k=1}^{N} a_k$

Then $s_{n+N} = \sum_{k=1}^{n+N} a_k = a_1 + a_2 + \dots + a_N + a_{N+1} + \dots + a_{n+N}$

$$= A + a_{1+N} + a_{2+N} + \dots + a_{n+N}$$

$$= A + \sum_{k=1}^{n} b_k$$

$$= A + s'_n$$

$$s_{n+N} = A + s'_n$$

ii) Define $s''_n = s_{n+N} \quad \forall n \in \mathbb{N}$.

$\Rightarrow \quad s''_n = A + s'_n$.

Using Algebra of limits, $\{s''_n\}$ converges iff $\{s'_n\}$ converges.

and $\lim_{n\to\infty} s''_n = A + \lim_{n\to\infty} s'_n$

Using exercise 2, $\{s''_n\} = \{s_{n+N}\}$ converges iff $\{s_n\}$ converges.

and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s''_n$.

$\Rightarrow \quad \{s_n\}$ converges $\Leftrightarrow$ $\{s'_n\}$ converges.

and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s''_n = \lim_{n\to\infty} s'_n + A$

$\Rightarrow \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty} s'_n + A$.

iii) $\sum_{k=1}^{\infty} a_k$ converges $\Leftrightarrow$ $\{s_n\}$ converges.

$\Leftrightarrow$ $\{s'_n\}$ converges (by (ii))

$\Leftrightarrow$ $\sum_{k=1}^{\infty} b_k$ converges.

Also $\sum_{k=1}^{\infty} a_k = A + \sum_{k=1}^{\infty} b_k$

4. 3.5.1.

$\sum_{k=1}^{\infty} a_k$ converges.

$\Rightarrow \quad a_k \to 0$ (by exercise 1).

$\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $|a_n| < \epsilon \quad \forall n \geq N$.

Using $\epsilon = 1$, get N s.t.

$$|a_n| < 1 \quad \forall n \geq N.$$

Since $\sum_{k=1}^{\infty} a_n$ has positive terms.

$$a_n = |a_n| < 1 \quad \forall n \geq N.$$

Therefore, $a_n^2 < a_n \quad \forall n \geq N$. (*).

Let $b_k = a_{k+N}$ and $c_k = a_{k+N}^2$

By exercise 3(iii)

$\sum a_k$ converges $\Rightarrow$ $\sum b_k$ converges.

Also $b_k \geq 0 \quad \forall k \in \mathbb{N}$.

$\Rightarrow s_n = \sum_{k=1}^{n} b_k$ is a

non decreasing convergent sequence.

$\Rightarrow \quad s_n \leq \sum_{k=1}^{\infty} b_k$

Also because of ⊛.

$$a_n^2 \leq a_n \quad \forall n \geq N.$$

$$\Rightarrow c_k = a_{k+N}^2 < a_{k+N} = b_k \quad \forall k \in \mathbb{N}.$$ (since $k+N \geq N$).

$$\Rightarrow s'_n = \sum_{k=1}^{n} c_k \leq \sum_{k=1}^{n} b_k = s_n$$

Also each $c_k \geq 0 \quad \forall k \in \mathbb{N}$.

$\Rightarrow s'_n$ is a non increasing sequence.

and $s'_n \leq s_n \leq \sum_{k=1}^{\infty} b_k$

So s'_n is a Monotone bounded sequence.

$\Rightarrow s'_n$ converges.

$\Rightarrow \sum_{k=1}^{\infty} c_k$ converges. $\Rightarrow \sum_{k=1}^{\infty} a_{k+N}^2$ converges

By exercise 3iii), $\sum_{k=1}^{\infty} a_k^2$ converges.

The converse does not hold:

$$a_k = \frac{1}{k}$$

$\sum a_k^2 = \sum \frac{1}{k^2}$ converges, but $\sum a_k = \sum \frac{1}{k}$ diverges.

5. 3.7.9.

Consider a series $\sum_{k=1}^{\infty} a_k$ that converges and has only finitely many negative terms.

Therefore, $\exists N \in \mathbb{N}$ s.t. $a_k \geq 0 \quad \forall k \geq N$. and all negative terms have index $< N$.

By exercise 3 iii), $\sum_{k=1}^{N} a_{k+N}$ converges.

Let $b_k = |a_k| \quad \forall k \in \mathbb{N}$.

Claim: We want to show $\sum b_k$ converges.

Pf: Observe that since $\forall k \geq N$. $a_k \geq 0$.

$$b_k = |a_k| = a_k$$

$$\Rightarrow \sum_{k=1}^{\infty} b_{k+N} = \sum_{k=1}^{\infty} a_{k+N} \text{ since}$$

$$\sum_{k=1}^{\infty} a_{k+N} \text{ converges.}$$

By exercise 3iii} $\sum b_k$ converges.

By the Claim $\sum b_k = \sum |a_k|$ converges.

Therefore $\sum a_k$ converges unconditionally. $\blacksquare$

~~So $\sum a_k$ can be safely rearran~~

3.7.10. Claim: The rearranged series converges nonabsolutely.

Pf: Let $\sum a_k$ be the nonabsolutely convergent series.

Let $\sigma : \mathbb{N} \to \mathbb{N}$ give the rearranged series

$$\sum_{k=1}^{\infty} a_{\sigma(k)}$$

Suppose to the contrary that $\sum_{k=1}^{\infty} a_{\sigma(k)}$ converges absolutely.

Let $b_k = |a_{\sigma(k)}|$

$\Rightarrow$ $\sum b_k$ converges absolutely (since $|b_k| = b_k$ $\forall k \in \mathbb{N}$.)

By Dirichlet's Theorem, any rearrangement of $\sum b_k$ converges.

Using $\sigma^{-1} : \mathbb{N} \to \mathbb{N}$.

$\Rightarrow$ $\sum_{k=1}^{\infty} b_{\sigma^{-1}(k)}$ converges.

$\Rightarrow$ $\sum_{k=1}^{\infty} |a_k|$ converges (since $b_k = |a_{\sigma(k)}|$ $\Rightarrow b_{\sigma^{-1}(k)} = |a_k|$)

$\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ converges absolutely contradicting the assumption that $\sum_{k=1}^{\infty} a_k$ converges nonabsolutely.