HW WEEK I Solutions

1. A.5.1. Prove $\sqrt{2}$ is irrational.

Proof: By Contradiction.

Assume $\sqrt{2}$ is rational. Therefore, $\exists\, m, n \in \mathbb{N}$ with $n \neq 0$ such that

$$\sqrt{2} = \frac{m}{n}$$

$$\Rightarrow \quad 2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$$

$$\Rightarrow \quad 2n^2 = m^2.$$

FACT: Every natural number has a unique prime factorization.

Since $2n^2$ is the same number as m^2

$2n^2$ has the same prime factorization as m^2.

Consider the exponent of 2 in the factorization.

Exponent of 2 in n^2 ~~...~~ is even: Let $n = 2^p n'$ where n' has a prime factorization without any 2's.

Thus $n^2 = (2^p n')^2 = 2^{2p} n'^2$

Thus exponent of 2 in n^2 is even. $(2p)$.

$\therefore$ Exponent of 2 in $2n^2 = 2 \cdot 2^{2p} n'^2 = 2^{2p+1} n'^2$ is odd.

But exponent of 2 in m^2 is even by the above argument. This is a contradiction $\blacksquare$

A.5.2. Show that there are infinitely many primes.

Proof: By contradiction.

Suppose there are finitely many primes
$$p_1, p_2, \ldots, p_n$$
Consider $N = p_1 \times p_2 \times \cdots \times p_n + 1$

This is a natural number.

FACT: Every natural number is divisible by some prime.

However, when N is divided by p_i it leaves remainder 1.

This is a contradiction.

A.6.1. Prove: x is irrational $\Rightarrow$ $x+r$ is irrational $\forall$ rational r.

Proof: By contrapositive.

To prove: $x+r$ is rational for some rational r $\Rightarrow$ x is rational

If $x+r$ is rational for some rational r.

$\exists\ m_1, n_1 \in \mathbb{N}$ and $m_2, n_2 \in \mathbb{N}$

s.t. $r = \dfrac{m_1}{n_1}$

and $x + \dfrac{m_1}{n_1} = \dfrac{m_2}{n_2}$

$$\Rightarrow x = \frac{m_2}{n_2} - \frac{m_1}{n_1} = \frac{m_2 n_1 - m_1 n_2}{n_2 n_1}$$

Thus x is rational. $\boxtimes$

A.7.3. $P \equiv$ "Every differentiable function is continuous

Converse of P

$\equiv$ Every continuous function is differentiable.

Contrapositive

$\equiv$ Every function that is NOT continuous is NOT differentiable.

Only the contrapositive is true., not the converse

A ~~counterexample to~~ the converse is $f(x) = |x|$.

A.8.1. $1^2 + 2^2 + 3^2 + \dots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Proof by induction on n.

1 Establish $P(1)$ is true :

$$1^2 = \frac{1(1+1)(2+1)}{6} = 1. \checkmark$$

2. ~~Assume~~ Show $P(n-1)$ ~~is true~~ $\Rightarrow P(n)$ $\forall\, n \geq 2$.

$$1^2 + 2^2 + 3^2 + \dots n^2$$

$$= \underbrace{1^2 + 2^2 + 3^2 + \dots (n-1)^2}\ + n^2$$

By the induction hypothesis,

$$1^2+2^2+\dots+(n-1)^2 = \frac{(n-1)(n-1+1)(2(n-1)+1)}{6}$$
$$= \frac{(n-1)(n)(2n-1)}{6}$$

Thus,

$$1^2+2^2+\dots n^2 = \frac{(n-1)n(2n-1)}{6} + n^2$$
$$= \frac{(n-1)n(2n-1) + 6n^2}{6}$$
$$= \frac{n[(n-1)(2n-1) + 6n]}{6}$$
$$= \frac{n[2n^2-3n+1+6n]}{6}$$
$$= \frac{n[2n^2+3n+1]}{6} = \frac{n(n+1)(2n+1)}{6}. \quad \blacksquare$$

A.8.2. For $n=1,2,3,4$ & 5 guess that the formula is

By induction $1+3+5+\dots+2n-1 = n^2$

1. Establish $P(1)$ is true: $1 = 1^2$ ✓.

2. Show $P(n-1) \Rightarrow P(n)$ $\forall n \geq 2$

$$1+3+5+\dots+2n-1$$
$$= 1+3+5+\dots 2(n-1)-1 + 2n-1$$

By the I.H.,

$$1+3+5+\dots+2(n-1)-1 = (n-1)^2.$$

Thus, $1+3+5+\dots 2(n-1)-1+2n-1$

$$= (n-1)^2 + 2n-1$$

$$= n^2-2n+1+2n-1$$

$$= \underline{\underline{n^2}}$$

A.8.4 Show $(1+x)^n \geq 1+nx \quad \forall\, n=1,2,\dots$ when $x>0$.

1. Establish $P(1)$ is true: $(1+x)^1 \geq 1+(1)x = 1+x$ ✓.

2. Show $P(n-1) \Rightarrow P(n) \quad \forall\, n \geq 2$.

$$(1+x)^n = (1+x)^{n-1}(1+x)$$

By I.H. $\boxed{(1+x)^{n-1} \geq 1+(n-1)x.}$

Since $x>0$, $1+x>0$

$\therefore$ Multiplying the boxed inequality by $(1+x)$: $(1+x)^{n-1}(1+x) \geq (1+(n-1)x)(1+x)$

$$\Rightarrow (1+x)^n \geq 1+(n-1)x+x+(n-1)x^2$$

$$= 1+nx+(n-1)x^2$$

$$\geq 1+nx$$

The last inequality follows because $n \geq 2$ and $x^2>0$ and so $(n-1)x^2>0$.

A.8.5 Prove : $1+r+r^2+\dots r^n = \dfrac{1-r^{n+1}}{1-r}$ $\forall n \in \mathbb{N}$.

for any real $r \neq 1$.

Proof By induction

1. Establish $P(1)$:

To show: $1+r = \dfrac{1-r^{1+1}}{1-r}$

The RHS is $\dfrac{1-r^2}{1-r} = \dfrac{\cancel{(1-r)}(1+r)}{\cancel{(1-r)}}$

which equals the LHS. ✓.

2. Show $P(n-1) \Rightarrow P(n)$ $\forall n \geq 2$.

$$1+r+r^2+\dots+r^n = 1+r+r^2+\dots+r^{n-1}+r^n$$

$$= \frac{1-r^{(n-1)+1}}{1-r} + r^n$$

(By I.H.)

$$= \frac{1-r^n}{1-r} + r^n$$

$$= \frac{1-r^n+r^n(1-r)}{1-r} = \frac{1-r^{n+1}}{1-r}$$

∎

A.8.9. ~~the way~~ The induction step
$$P(n) \Rightarrow P(n+1)$$
should hold for $\forall n \geq 1$.

However for $n = 1$, the argument breaks down.

Consider ~~when~~ $P(n+1) = P(2)$.

So we have 2 birds B_1, B_2.

If we take out B_2 in our hand, B_1 is left and of course it is one species.

But when we take B_1 and put B_2 back in, B_2 ~~cannot~~ cannot be compared to anyone → no one is left!

So the induction step <u>does not</u> work for <u>$n=1$</u>

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2. Show x^2 is even $\Leftrightarrow$ x is even.

Proof: First, show x^2 is even $\Rightarrow$ x is even.

By contrapositive, we show
$$x \text{ is odd} \Rightarrow x^2 \text{ is odd}.$$

If x is odd $\Rightarrow \exists$ natural number n such that $x = 2n+1$.

$$\therefore x^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2+2n)+1$$

So x^2 is odd.

Now show x is even $\Rightarrow$ x^2 is even.

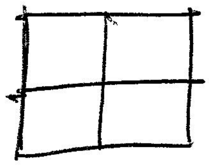
If x is even, $\exists\, m \in \mathbb{N}$ s.t. $x = 2m$.

$\therefore\ x^2 = (2m)^2 = 4m^2 = 2(2m^2)$

$\therefore\ x^2$ is even. ∎

3. Proof by induction:

1. Establish $P(1)$:



It is clear no matter which square is marked, we can tile:

2. $P(n-1) \Rightarrow P(n)$.

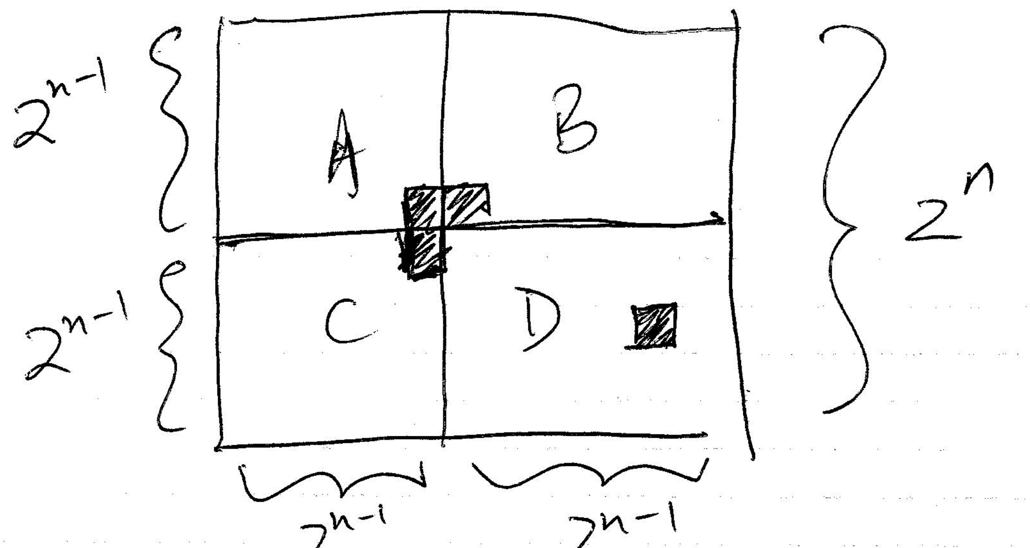


Fig 1

Divide the $2^n \times 2^n$ board into
4 boards A, B, C, D. of
size $2^{n-1} \times 2^{n-1}$. as shown in Fig. 1.

The original marked square is in one of the ~~squa~~ smaller boards A, B, C or D.

We show the argument when it is in D and the other cases are the same.

By I.H. A can be tiled with the bottom ~~left~~ right square marked.

B can be tiled with bottom left square marked.

C " " " " top right square marked.

Finally a ~~~~ domino to cover the marked squares from A, B + C.

$\boxtimes$

4. Proof by contrapositive. ~~[illegible]~~

To show: $f(t) = 0 \quad \forall\, 0 \le t \le 1 \Rightarrow \int_0^1 f(t)\,dt = 0.$

This is true because $\int_0^1 0\,dt = 0.$

5. Statement $P \equiv$ x is div. by $4 \Rightarrow x$ is div. by 2.

Contrapositive : x is not div. by $2 \Rightarrow x$ is not div. by 4.

Converse : x is div. by $2 \Rightarrow x$ is div. by 4.

Contrapositive of Converse : x is NOT div. by 4 $\Rightarrow x$ is NOT div. by 2.

Statement P is true : Proof : x is div by 4

$\Rightarrow x = 4n$ for some $n \in \mathbb{N}$

$\Rightarrow x = 2(2n)$

$\Rightarrow x$ is div. by 2.

$\therefore$ Contrapositive of P is true.

Converse of P is false : 6 is a counterexample

Contrapositive of the converse is false : 6 " " ".

A.9.1.

a) False.
b) True.
c) True
d) False
e) True.
f) False.
g) True.

A.9.2.

a) $\exists x \in \mathbb{R}, \quad x < 0.$
b) $\forall x \in \mathbb{R}, \quad x < 0.$
c) $\exists x \in \mathbb{R}, \quad x^2 < 0$
d) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad x + y \neq 1.$
e) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad x + y \neq 1.$
f) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad x + y \neq 1.$
g) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad x + y \neq 1.$

7. <u>Use strong induction</u>:

1. Establish $P(1)$, $P(2)$ is true:

$$P(1): a_1 = 1 \text{ and } \frac{\psi - \bar{\psi}}{\psi - \bar{\psi}} = 1. \checkmark$$

$$P(2): a_2 = 1 \text{ and } \frac{\psi^2 - \bar{\psi}^2}{\psi - \bar{\psi}} = \frac{(\psi + \bar{\psi})(\psi - \bar{\psi})}{\psi - \bar{\psi}}$$
$$= \psi + \bar{\psi}$$
$$= 1 \checkmark$$

2. Assume $P(1) \dots P(n-1)$ are all true, and show $P(n)$ is true. $\forall n \geq 3$

Since $n \geq 3$,

$$a_n = a_{n-1} + a_{n-2}$$
$$= \frac{\psi^{n-1} + \bar{\psi}^{n-1}}{\psi - \bar{\psi}} + \frac{\psi^{n-2} + \bar{\psi}^{n-2}}{\psi - \bar{\psi}} \quad \left[\text{By I.H.}\right]$$
$$= \frac{(\psi^{n-1} + \psi^{n-2}) + (\bar{\psi}^{n-1} + \bar{\psi}^{n-2})}{\psi - \bar{\psi}}$$

Now we prove $\psi^{n-1} + \psi^{n-2} = \psi^n$ and similarly for $\bar{\psi}$.

Since ψ is a root of $x^2 = x + 1$ we have $\psi^2 = \psi + 1$

$$\Rightarrow \quad \psi^{n-2} \times (\psi^2 = \psi + 1)$$

$$\Rightarrow \psi^n = \psi^{n-1} + \psi^{n-2}. \quad (\text{similarly for } \bar{\psi})$$

$$\therefore \quad a_n = \frac{(\psi^{n-1} + \psi^{n-2}) + (\bar{\psi}^{n-1} + \bar{\psi}^{n-2})}{\psi - \bar{\psi}}$$

$$= \boxed{\frac{\psi^n - \bar{\psi}^n}{\psi - \bar{\psi}}}$$