Practice Midterm II
MAT-165 Mathematics and Computers

Fall 2012

Name ____

- **This test is closed notes, closed book.**
- **There are 12 pages and 8 questions total.**
- **The maximum score in the test is 130 points.**
- **IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY'S HONOR CODE TO HAVE ANOTHER PERSON TAKE YOUR EXAM FOR YOU.**

Signature ____

Problem	Score	Max Possible
1		15
2		40
3		30
4		10
5		7
6		8
7		10
8		10
Total		130

1. (a) **(5 pts)** State Sturm's Theorem for finding the number of distinct real roots of a polynomial in an interval.

   (b) **(5 pts)** State the Gordan-Disckon Lemma for monomial ideals.

(c) **(5 pts)** State Hilbert's Nullstellensatz.

2. Recall the k-coloring problem for graphs. Let $G = (V, E)$ be a graph where $V = \{1, 2, 3\}$ and $E = \{(1, 2), (2, 3), (1, 3)\}$ - the so called triangle graph.

   (a) **(10 pts)** Set up the system of polynomial equations which test if the graph G is 3 colorable.

(b) **(10 pts)** Set up the system of linear equations to test if there exists a Hilbert Nullstellensatz certificate of infeasibility of degree 1 for the system of polynomial equations from 2(a).

(c) **(10 pts)** Without explicitly solving the system from 2(b), decide whether the system has a solution. Justify your answer.

(d) **(5 pts)** Write down the adjacency matrix of the graph G.

(e) **(5 pts)** If we want to count the number of automorphisms of G using a system of polynomial equations, how many variables will the polynomials be defined over ?

3. You can use MAPLE for the following problems. Clearly state which functions you used, the output given by MAPLE and how that led to your conclusion. Here are a few MAPLE functions that you may find useful : In the *with*(*Groebner*) package, *Basis*() finds a (reduced) Groebner basis, *NormalForm*() finds the remainder after applying multivariate division algorithm, *solve*() finds the solutions to a system of polynomial equations. In the *with*(*LinearAlgebra*) package, *Determinant*() computes the determinant of a matrix.

   (a) **(10 pts)** Is the polynomial $p = x^3z - 2y^2$ in the ideal

   $$I = \langle xz - y, xy + 2z^2, y - z \rangle?$$

   (b) **(10 pts)** Maximize the value of $x^3+2xyz-z^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

(c) **(10 pts)** Consider the polynomials $f, g \in \mathbb{K}[x, y]$ given by $f = x^2y + x - 1$ and $g = x^2y + x + y^2 - 4$. Write down the Sylvester matrix $Syl(f, g, x)$. Use MAPLE to compute the resultant $Res(f, g, x)$.

4. **(10 pts)** Prove the Hilbert Nullstellensatz for univariate polynomials: Let $\mathbb{K}$ be an algebraically closed field. Suppose $f_1, \ldots, f_k \in \mathbb{K}[x]$. Show that $f_1, \ldots, f_k$ have no common roots in $\mathbb{K}$ if and only if there exist polynomials $h_1, \ldots, h_k \in \mathbb{K}[x]$ such that $1 = h_1 f_1 + h_2 f_2 + \ldots + h_k f_k$. (Simply quoting the Hilbert Nullstellensatz for general polynomials will not get any credit. In fact, the proof you give here is needed to prove the Base Case in the induction done in class for the general Hilbert Nullstellensatz.)

5. **(7 pts)** Consider any ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Recall the definition of the l-th elimination ideal $I_l = I \cap \mathbb{K}[x_l, \ldots, x_n]$. Show that I_l is an ideal of $\mathbb{K}[x_l, \ldots, x_n]$ for every $l = 1, \ldots, n$.

6. **(8 pts)** Let $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal and let $f \in \mathbb{K}[x_1, \ldots, x_n]$ be any polynomial. Recall that $LM(I) = \{LM(p) : p \in I\}$ is the set of all leading monomials of polynomials in I (note that $LM(I)$ is not an ideal). Show that f can be expressed as $f = g + r$ such that $g \in I$ and no term of r is divisible by an element in $LM(I)$.

7. **(10 pts)** Given two polynomials $f, g \in \mathbb{K}[x_1, \ldots, x_n]$ how would you find a polynomial $h \in \langle f, g \rangle$ such that the remainder when h is divided by (f, g) is nonzero, or show that no such h exists ? Clearly justify your answer.

8. **(10 pts)** Let $f = x^3 - 3x^2 - 4$ and $g = -3x^2 + x + 10$. Prove that there exist nonzero polynomials $A, B \in \mathbb{R}[x]$ such that $Af + Bg = 0$ (you may find the $gcd()$ command in MAPLE useful). Next, explain how you can solve a linear system of equations to find these polynomials (you DO NOT have to solve this linear system, only set it up). Clearly justify all your answers.