Homework Problems

1. The *Division Algorithm* for univariate polynomials $f, g \in \mathbb{K}[x]$ finds $q, r \in \mathbb{K}[x]$, with $\deg(r) < \deg(g)$, such that $f = q \cdot g + r$. Prove that this pair q, r is unique, i.e., if $q', r' \in \mathbb{K}[x]$, with $\deg(r') < \deg(g)$, such that $f = q' \cdot g + r'$, then $q = q'$ and $r = r'$.

2. Prove that iterative version of the Division Algorithm discussed in class actually terminates without getting into an infinite loop.

3. Do Problem 2 on page 46 of the textbook.

4. Do Problem 13 on page 47 of the textbook.

5. Show that there are $\binom{n+d-1}{d}$ monomials of degree d over n variables.

6. Let $f_1, \ldots, f_k \in \mathbb{K}[x]$ be univariate polynomials.

   (i) Prove that
   $$GCD(f_1, \ldots, f_k) = GCD(f_1, GCD(f_2, \ldots, f_k)).$$
   How can this be used to give an algorithm for finding $GCD(f_1, \ldots, f_k)$ using the GCD algorithm for two polynomials ?

   (ii) Prove that there exist polynomials $h_1, \ldots, h_k \in \mathbb{K}[x]$ such that $GCD(f_1, \ldots, f_k) = \sum_{i=1}^{k} h_i f_i$.

7. Use the *quo*() and *rem*() commands in MAPLE as a tool to compute the following GCD's. You should not run a GCD code on MAPLE - this exercise is meant to give you practice with working out the GCD algorithm by hand: MAPLE is just to help you compute quotients and remainders.

   (i) $GCD(x^6 - 1, x^4 - 1)$.

   (ii) $GCD(x^3 + 2x^2 - x - 2, x^3 + 2x^2 - x + 2)$

   (ii) $GCD(x^4 + x^2 + 1, x^4 - x^2 - 2x - 1, x^3 - 1)$.

8. Consider $f_1, \ldots, f_k \in \mathbb{C}[x]$, i.e., polynomials with complex coefficients. Prove that the set of common roots (in $\mathbb{C}$) is empty if and only if their GCD is 1.

9. Use MAPLE to find all the real roots of the polynomial $3x - 25x^3 + 60x - 20$.

10. Explain a method you can use to decide whether a univariate polynomial has roots of multiplicity more than 1 (i.e., repeated roots). Use it to test whether $x^3 - 2x^2 - 4x + 8$ has multiple roots.

11. Using Descartes' rule of signs find as much information as you can about the possible number of real roots (counting multiplicities) for each of the following polynomials (You can use MAPLE to compute GCD's and quotients and remainders):

    (a) $x^4 - x^2 + x - 2$

    (b) $x^9 - x^5 + x^2 + 2$

    (c) $x^5 + 2x^3 - x^2 + x - 1$

12. Apply Sturm's sequences and find out exactly the number of distinct roots for each of the polynomials of previous problem (You can use MAPLE to compute GCD's and quotients and remainders).

13. Prove the assertion from class : If $V = V(f_1, \ldots, f_k)$ and $W = V(g_1, \ldots, g_s)$, then $V \cup W = V(\{f_i g_j : i = 1, \ldots, k; j = 1, \ldots, s\})$. (A complete proof appears on page 11 of the text)

14. Sketch (or visualize) the following affine varieties (or at least the real parts of it!) in $\mathbb{R}^2$:
    (a) $V(x^2 - y^2)$
    (b) $V(x^2 + 4y^2 + 2x - 16y + 1)$
    and in $\mathbb{R}^3$:
    (c) $V(xz^2 - xy)$
    (d) $V(x^2 + y^2 + z^2 - 1, x^2 + y^2 + (z-1)^2 - 1)$

15. Do Problem 2, 3, 5 and 6 on page 12 of textbook.

    Recall the definition of the ideal generated by a finite set of polynomials : Given $f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_n]$,

    $\langle f_1, \ldots, f_k \rangle = \{p \in \mathbb{K}[x_1, \ldots, x_n] : p = h_1 f_1 + h_2 f_2 + \ldots + h_k f_k \text{ for some } h_1, \ldots, h_k \in \mathbb{K}[x_1, \ldots, x_n]\}.$

    Another way to think of this ideal is that it is the set of all "polynomial combinations" of $f_1, \ldots, f_k$.

16. Do Problems 2, 3, 5 on page 36 of the textbook.

17. **Ideals in Univariate Polynomials** In this problem, we characterize ideals in univariate polynomials.
    (a) Let $f_1, \ldots, f_k \in \mathbb{K}[x]$. Show that

    $$\langle f_1, \ldots, f_k \rangle = \langle GCD(f_1, \ldots, f_k) \rangle.$$

    (Hint: Use Problem 6, part (ii) above.) Thus for univariate polynomials, an ideal generated by a finite number of polynomials is actually generated by a *single* polynomial ! In the next part we further strengthen this assertion.

    (b) Let $I \subseteq \mathbb{K}[x]$ be *any* ideal in $\mathbb{K}[x]$ (note that we are not assuming that it is generated by a finite set of polynomials as in part (a).) Show that there exists $f \in \mathbb{K}[x]$ such that $I = \langle f \rangle$. (Hint: Consider the smallest degree polynomial in $\mathbb{K}[x]$.)

    Because of the above property, the univariate polynomials are called a *principal ideal domain (PID)*. See also Corollary 4 in the textbook.

    (c) Part (a) suggests an algorithm for testing if a given $f \in \mathbb{K}[x]$ belongs to the ideal $\langle f_1, \ldots, f_k \rangle$, without having to "guess" at the appropriate combinations like we had to for Problem 15. Use this algorithm to test if the polynomial $x^2 - 4$ is in the ideal generated by the polynomials $\langle x^3 + x^2 - 4x - 4, x^3 - x^2 - 4x + 4, x^3 - 2x^2 - x + 2 \rangle$.

18. Do Problem 1 from page 52 of the textbook.

19. The basis of an ideal is different from a basis in linear algebra in that there is no concept analogous to linear independence. As a consequence when we write an element f in the ideal $\langle f_1, \ldots, f_k \rangle$ as $f = \sum_{i=1}^{k} h_i f_i$ the coefficients h_i are not always unique. As an example, write $x^2 + xy + y^2 \in \langle x, y \rangle$ in two different ways.

20. Do Problems 7, 8 on page 37 of the textbook.

21. Explain a method you can use to decide whether a univariate polynomial has roots of high multiplicity. Use it to test whether $x^3 - 2x^2 - 4x + 8$ has multiple roots. You can use MAPLE for manipulating univariate polynomials (like GCD, quotients, remainders etc.)

22. True or false: The set of polynomials $p \in \mathbb{R}[x, y, z]$ such that $p(t^2, t^3, t^4) = 0$ for all $t \in \mathbb{R}$ is an ideal.

23. Is x^2 an element inside the ideal $\langle x - y^2, xy \rangle$? Is $\langle x - y^2, xy \rangle = \langle x^2, xy \rangle$? Why or why not? If not, is any one of them included in the other ?

24. Do Problems 1, 2, 3, 7, 10, 11 on page 60 of the textbook.

25. Let $>$ be a total order on the monomials $\mathbb{Z}_+^n$ such that $\alpha > \beta \Rightarrow \alpha + \gamma > \beta + \gamma$. Show that $>$ is a well-ordering *if and only if* $\alpha > 0$ for all $\alpha \in \mathbb{Z}_+^n \setminus \{0\}$. Why does this immediately show that *graded lex* and *graded reverse lex* are monomial orderings ? (Hint: For the *if* direction, use the Gordan-Dickson Lemma)

26. Suppose we use the multivariate division algorithm for dividing f by $f_1, \ldots, f_k$, and we get two different remainders r_1 and r_2 corresponding to two different orders that we pick on $f_1, \ldots, f_k$. Is $r_1 - r_2 \in \langle f_1, \ldots, f_k \rangle$? If so, how would you find polynomials $h_1, \ldots, h_k$ such that $r_1 - r_2 = h_1 f_1 + \ldots + h_k f_k$ from the output of the multivariate division algorithm ?

27. Do problems 1, 2, 6, 7, 9 on page 68 of the textbook.

28. Do problems 1, 3, 4, 5 on page 74 of the textbook.

29. Recall that we can define an ideal generated by infinitely many polynomials : Given any set $F \subseteq \mathbb{K}[x_1, \ldots, x_n]$ of polynomials (possibly infinite), define

$$I = \langle F \rangle = \{ \sum_{\text{finite sum}} h_\alpha f_\alpha : h_\alpha \in \mathbb{K}[x_1, \ldots, x_m], f_\alpha \in F \}$$

Suppose $F = \{f_1, f_2, f_3, \ldots\}$ is an infinite sequence of polynomials. Let $I = \langle F \rangle$. Show that there exists a natural number $N \in \mathbb{N}$ such that $I = \langle f_1, \ldots, f_N \rangle$. (Be careful : the Hilbert Basis theorem shows that I is finitely generated; however, it does not give any guarantee that the basis will come from the elements in F).

30. We defined Gröbner bases in terms of leading monomials ($\langle LM(I) \rangle = \langle LM(g_1), \ldots, LM(g_s) \rangle$). We can achieve the same by using *leading terms* instead (recall that these are different from the leading monomials - they include the coefficients). Suppose I is an ideal. Let $LT(I)$ denote the set of all leading *terms* of polynomials in I, i.e., $LT(I) = \{LT(f) : f \in I\}$. Of course, $LT(I)$ is a set of polynomials and therefore one can define $\langle LT(I) \rangle$ as the ideal generated by all of these (infinitely many) polynomials (see exercise 29). Suppose $g_1, \ldots, g_s \in I$ such that $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_s) \rangle$. Show that $\{g_1, \ldots, g_s\}$ is a Gröbner basis for I.

31. Show that for each natural number $n \geq 1$ there exists a monomial ideal $I \subseteq \mathbb{K}[x, y]$ such that every basis of I has at least n elements.

32. Do Problems 1, 2, 3, 4, 5, 6, 7, 9a, 9c, 11 from pages 87-88 of the textbook.

33. Recall that given an ideal I, we define the variety of this ideal $V(I) = \{a \in \mathbb{K}^n : f(a) = 0 \text{ for all } f \in I\}$. Show that if $I = \langle f_1, \ldots, f_k \rangle$, $V(I) = V(f_1, \ldots, f_k)$.

34. For the following two problems, use MAPLE to do your calculations. You can compute S-polynomials and remainders using the commands $SPolynomial()$ and $NormalForm()$ respectively, from the package $with(Groebner)$.

- Do Problems 2, 3, 5 from page 94 of the textbook.
- Do Problems 1, 2, 3, 4, 5a.
- Find the maximum of $x^2 + y^2 + xy$ subject to $x^2 + 2y^2 = 1$.
- For the following systems of polynomial equations answer the following questions: Is the system solvable? Does it have a finite number of solutions? If not, determine the dimension of the solution space.
  (a) $x^2 - 2x + 5, xy^2 + yz^3, 3y^2 - 8z^3$.
  (b) $x^2z^2 + x^3, xz^4 + 2x^2z^2 + x^3, y^2z - 2yz^2 + z^3$.
- Find the common zeros of the polynomials $xyz - w, yzw - x, zwx - y, xyw - z$.

35. Consider any ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Recall the definition of the l-th elimination ideal $I_l = I \cap \mathbb{K}[x_l, \ldots, x_n]$. Show that I_l is an ideal of $\mathbb{K}[x_l, \ldots, x_n]$ for every $l = 1, \ldots, n$.

36. Recall that we made a special change of variables during the proof of Hilbert's Nullstellensatz:

$$\begin{aligned} x_1 &= \tilde{x}_1 \\ x_2 &= \tilde{x}_2 + a_2\tilde{x}_1 \\ &\vdots \\ x_n &= \tilde{x}_n + a_n\tilde{x}_1 \end{aligned}$$

Under this transformation, every polynomial $f(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$ transforms into $\tilde{f}(\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{K}[\tilde{x}_1, \ldots, \tilde{x}_n]$. Consider any ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Let $\tilde{I} \subseteq \mathbb{K}[\tilde{x}_1, \ldots, \tilde{x}_n]$ be the transformed polynomials in I. Show that $\tilde{I}$ is an ideal of $\mathbb{K}[\tilde{x}_1, \ldots, \tilde{x}_n]$.

37. Do Problem 7 from page 174 of the textbook.

38. Do Problems 12, 13 and 15 from page 160 of the textbook.

39. Do Problems 1, 2a, 2b, 8, 10 from page 166 of the textbook.

40. Consider an ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$. Let $c_2, \ldots, c_n$ be elements of $\mathbb{K}$. Now consider the set of univariate polynomials in x_1 given by $\bar{I} = \{f(x_1, c_2, \ldots, c_n) : f \in I\}$. Show that $\bar{I}$ is an ideal of $\mathbb{K}[x_1]$.