We will be refering to the following relaxation of an MILP discussed in class :

$$\begin{aligned} x &= f + \sum_{j=1}^{k} r^j s_j \\ x &\in \mathbb{Z}^q \\ s &\in \mathbb{R}^k_+ . \end{aligned} \tag{1}$$

1. **Orthogonality of basis vectors for a lattice** Consider the lattice of $\mathbb{R}^n$ generated by $b_1 = (1,0), b_2 = (\frac{1}{2}, 1)$. Show that this lattice cannot have an orthogonal basis.

2. **Claims left over from the Lecture 3**

   a) Consider a closed, convex set C with a point f in its interior. Recall the Minkowski functional defined in class : $\psi_C(r) = \inf\{t > 0 \,|\, f + \frac{r}{t} \in C\}$. Show that ψ_C is positively homogeneous (i.e. $\psi(tr) = t\psi(r)$ for any real number $t \geq 0$ and $r \in \mathbb{R}^n$) and that $C = \{x \,|\, \psi_C(x - f) \leq 1\}$. (We showed in class that it is subadditive and that using these three properties we can use the Minkowski functional to derive valid inequalities).

   b) Consider two closed convex sets C_1 and C_2 containing f in their interiors, such that $C_1 \subseteq C_2$. Show that the associated Minkowski functionals satisfy the inequality $\psi_{C_2}(r) \leq \psi_{C_1}(r)$ for all $r \in \mathbb{R}^n$. (This fact implies that we only need to consider *maximal* lattice-free convex sets to derive our valid inequalities).

   c) We argued in class that the recession cone of the system (1) is $\mathbb{R}^k_+$, assuming rational data. Show that this implies that *any* valid inequality $\sum_{j=1}^{k} \gamma_j s_j \geq \alpha$ for (1) has the property that $\gamma_j \geq 0$ for all $j \in \{1, \ldots, k\}$. Hence, we can assume that non-trivial valid inequalities take the form $\sum_{j=1}^{k} \gamma_j s_j \geq 1$. Such polyhedra are called *blocking polyhedra*.

3. We want to analyze the formula given in class for deriving cutting planes for (1) from maximal lattice-free convex set. Let B be a maximal lattice-free convex set, with f in its interior

   a) Show that B can be expressed by a set of inequalities of the form $\{x \in \mathbb{R}^n \,|\, a_i(x - f) \leq 1, i \in I\}$, where I is some finite index set.

   b) Show that the inequality $\sum_{j=1}^{k} \psi(r^j) s_j \geq 1$ is valid for (1) where

   $$\psi(r) = \max_{i \in I} a_i \cdot r$$

   Do not use the fact that this formula is equal to the Minkowski functional. Instead directly prove that the above ψ is subadditive (i.e. $\psi(r_1 + r_2) \leq \psi(r_1) + \psi(r_2)$), positively homogeneous and that $B = \{x \,|\, \psi(x - f) \leq 1\}$. Using the same proof idea as in Lecture 3 these three properties imply that $\sum_{j=1}^{k} \psi(r^j) s_j \geq 1$ is valid for (1).

4. Consider any closed, convex, full-dimensional polyhedron B whose recession cone is *not* full-dimensional. Let f be an interior point of B. Show that the function defined above $\psi(r) = \max_{i \in I} a_i \cdot r$ is equal to

the Minkowski functional of B, i.e. $\psi(r) = \inf\{t > 0 \mid f + \frac{r}{t} \in B\}$. Note that maximal lattice-free convex sets do not have full dimensional recession cones and so this justifies the use of this formula to compute the Minkowski functional when deriving cutting planes for (1).

5. We consider a generalization of Problem 5 from Homework 1. We needed this generalization for our proof of the structure of maximal lattice-free convex sets (in particular, when we proved that the lineality space and recession cone coincide). Consider a lattice Λ of $\mathbb{R}^n$. Let $y \in \Lambda$ be a lattice point and $r \in \mathbb{R}^n$ be any vector. Show that given any ϵ and $\bar{\lambda} \geq 0$, there exists a $\lambda \geq \bar{\lambda}$ and a lattice point $p \in \Lambda$ such that $\|p - (y + \lambda r)\| \leq \epsilon$. In other words, for any ϵ, there exists a lattice point that has distance at most ϵ from the half-line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. You can assume the result of Problem 5 from Homework 1 holds, without proof.

6. Do Exercises 6.5, 6.6, 6.12, 6.13 from the notes on IP handed out during Mini 3.

7. Recall the Reduction Algorithm from class for deciding the facets of the two-row version of (1). Suppose that for a Type 3 triangle, 0 rays survive at the end of the Reduction Algorithm. Show that this means that the problem only had rays pointing to the integer points in the sides of the triangle.

8. **Lineality space of MLFCs** Recall that the lineality space L of a maximal lattice-free convex set B was claimed to be a lattice-subspace in class. I only gave a very high level idea of the proof. Your task is to fill in the technical details of this claim. Recall that the main idea is to try and project everything (the lattice and B) onto the orthogonal complement of the lineality space L. If L is not a lattice-subspace we will arrive at a contradiction by analyzing what happens in the projection. First show that there is a subspace $U \neq \{0\}$ of $L^\perp$ (the orthogonal complement of L) such that the projected lattice is dense (with respect to the Euclidean norm) in this subspace. Then argue that if $U = L^\perp$, then there is a contradiction with the fact that B is lattice-free. Otherwise, show that L can be extended to a larger linear space using the maximality of B, hence contradicting the fact that L is the lineality space of B.