Outline

1. Generic LineSearch Framework

2. Computing a descent direction $p_k$ (search direction)
   - Steepest descent direction
   - Modified Newton direction
   - Quasi-Newton directions for medium scale problems
   - Limited-memory quasi-Newton directions for large-scale problems
   - Linear CG method for large-scale problems

3. Choosing the step length $\alpha_k$ (linesearch)
   - Backtracking-Armijo linesearch
   - Wolfe conditions
   - Strong Wolfe conditions

4. Complete Algorithms
   - Steepest descent backtracking Armijo linesearch method
   - Modified Newton backtracking-Armijo linesearch method
   - Modified Newton linesearch method based on the Wolfe conditions
   - Quasi-Newton linesearch method based on the Wolfe conditions
The problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad x \in \mathbb{R}^n
\end{align*}
\]

where the objective function \( f : \mathbb{R}^n \to \mathbb{R} \)

- assume that \( f \in C^1 \) (sometimes \( C^2 \)) and is Lipschitz continuous
- in practice this assumption may be violated, but the algorithms we develop may still work
- in practice it is very rare to be able to provide an explicit minimizer
- we consider iterative methods: given starting guess \( x_0 \), generate sequence
  \[ \{x_k\} \quad \text{for} \quad k = 1, 2, \ldots \]
- \textbf{AIM:} ensure that (a subsequence) has some favorable limiting properties
  - satisfies first-order necessary conditions
  - satisfies second-order necessary conditions

\textbf{Notation:} \( f_k = f(x_k) \), \( g_k = g(x_k) \), \( H_k = H(x_k) \)

The basic idea

- consider descent methods, i.e., \( f(x_{k+1}) < f(x_k) \)
- calculate a search direction \( p_k \) from \( x_k \)
- ensure that this direction is a descent direction, i.e.,
  \[ g_k^T p_k < 0 \quad \text{if} \quad g_k \neq 0 \]
  so that, for small steps along \( p_k \), the objective function \( f \) will be reduced
- calculate a suitable steplength \( \alpha_k > 0 \) so that
  \[ f(x_k + \alpha_k p_k) < f_k \]
- computation of \( \alpha_k \) is the linesearch
- the computation of \( \alpha_k \) may itself require an iterative procedure
- generic update for linesearch methods is given by
  \[ x_{k+1} = x_k + \alpha_k p_k \]
Issue 1: steps might be too long

Figure: The objective function $f(x) = x^2$ and the iterates $x_{k+1} = x_k + \alpha_k p_k$ generated by the descent directions $p_k = (-1)^{k+1}$ and steps $\alpha_k = 2 + 3/2^{k+1}$ from $x_0 = 2$.

- decrease in $f$ is not proportional to the size of the directional derivative!

Issue 2: steps might be too short

Figure: The objective function $f(x) = x^2$ and the iterates $x_{k+1} = x_k + \alpha_k p_k$ generated by the descent directions $p_k = -1$ and steps $\alpha_k = 1/2^{k+1}$ from $x_0 = 2$.

- decrease in $f$ is not proportional to the size of the directional derivative!
What is a practical linesearch?

● in the “early” days exact linesearchs were used, i.e., pick \( \alpha_k \) to minimize
  \[
  f(x_k + \alpha p_k)
  \]
  ▶ an exact linesearch requires univariate minimization
  ▶ cheap if \( f \) is simple, e.g., a quadratic function
  ▶ generally very expensive and not cost effective
  ▶ exact linesearch may not be much better than an approximate linesearch

● modern methods use inexact linesearchs
  ▶ ensure steps are neither too long nor too short
  ▶ make sure that the decrease in \( f \) is proportional to the directional derivative
  ▶ try to pick “appropriate” initial stepsize for fast convergence
    * related to how the search direction \( s_k \) is computed

● the descent direction (search direction) is also important
  ▶ “bad” directions may not converge at all
  ▶ more typically, “bad” directions may converge very slowly

Definition 2.1 (Steepest descent direction)

For a differentiable function \( f \), the search direction

\[
p_k \equiv -\nabla f(x_k) \equiv -g_k
\]

is called the steepest-descent direction.

● \( p_k \) is a descent direction provided \( g_k \neq 0 \)
  ▶ \( g_k^T p_k = -g_k^T g_k = -\|g_k\|^2 < 0 \)
● \( p_k \) solves the problem
  \[
  \minimize_{p \in \mathbb{R}^n} m^f_k(x_k + p) \ \text{subject to} \ \|p\|_2 = \|g_k\|_2
  \]
  ▶ \( m^f_k(x_k + p) \equiv f_k + g_k^T p 
  \]
  ▶ \( m^f_k(x_k + p) \approx f(x_k + p) 

Any method that uses the steepest-descent direction is a method of steepest descent.
Recall: at the point $x_k$, the steepest descent direction $p_k = -g_k$ is the solution of

$$\minimize_{p \in \mathbb{R}^m} \ m_k^*(x_k + p) = f_k + g_k^T p \quad \text{subject to} \quad ||p||_2 = ||g_k||_2$$

Observation: the steepest descent direction is also the unique solution to

$$\minimize_{p \in \mathbb{R}^n} \ f_k + g_k^T p + \frac{1}{2} p^T B_k p$$

- approximates second-derivative Hessian by the identify matrix $I$
- how often is this a good idea?
- is it a surprise that convergence is typically very slow?

Idea: choose positive definite $B_k$ and compute search direction as the unique minimizer

$$p_k = \arg\min_{p \in \mathbb{R}^n} \ m_k^Q(p) \overset{\text{def}}{=} f_k + g_k^T p + \frac{1}{2} p^T B_k p$$

- $p_k$ satisfies $B_k p_k = -g_k$
- why must $B_k$ be positive definite?
  - $B_k > 0 \implies m_k^Q$ is strictly convex $\implies$ unique solution
  - if $g_k \neq 0$, then $p_k \neq 0$ and is a descent direction
- $m_k^Q g_k = -p_k^T B_k g_k < 0 \implies p_k$ is a descent direction
- pick "intelligent" $B_k$ that has "useful" curvature information
- IF $H_k > 0$ and we choose $B_k = H_k$, then $s_k$ is the Newton direction. Awesome!

Question: How do we choose the positive-definite matrix $B_k$?

Ideally, $B_k$ is chosen such that
- $||B_k - H_k||$ is “small”
- $B_k = H_k$ when $H_k$ is “sufficiently” positive definite.

Comments:
- for the remainder of this section, we omit the suffix $k$ and write $H = H_k$, $B = B_k$, and $g = g_k$.
- for a symmetric matrix $A \in \mathbb{R}^{n \times n}$, we use the matrix norm

$$||A||_2 = \max_{1 \leq i \leq n} |\lambda_i|$$

with $\{\lambda_i\}$ the eigenvalues of $A$.
- the spectral decomposition of $H$ is given by $H = V \Lambda V^T$, where
  $$V = (v_1 \ v_2 \ \cdots \ v_n) \quad \text{and} \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$
  with $H v_j = \lambda_j v_j$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.
- $H$ is positive definite if and only if $\lambda_j > 0$ for all $j$.
- computing the spectral decomposition is, generally, very expensive!
Method 1: small scale problems \((n < 1000)\)

**Algorithm 1** Compute modified Newton matrix \(B\) from \(H\)

1. **input** \(H\)
2. Choose \(\beta > 1\), the desired bound on the condition number of \(B\).
3. Compute the spectral decomposition \(H = V \Lambda V^T\).
4. **if** \(H = 0\) **then**
   5. Set \(\varepsilon = 1\)
   6. **else**
5. Set \(\varepsilon = \|H\|_2/\beta > 0\)
8. **end if**
9. Compute 
   \[ \bar{\Lambda} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n) \quad \text{with} \quad \bar{\lambda}_j = \begin{cases} \lambda_j & \text{if } \lambda_j \geq \varepsilon \\ \varepsilon & \text{otherwise} \end{cases} \]
10. **return** \(B = V \bar{\Lambda} V^T \succ 0\), which satisfies \(\text{cond}(B) \leq \beta\)

- replaces the eigenvalues of \(H\) that are “not positive enough” with \(\varepsilon\)
- \(A = \Lambda + D\), where
  \[ D = \text{diag} \left( \max(0, \varepsilon - \lambda_1), \max(0, \varepsilon - \lambda_2), \ldots, \max(0, \varepsilon - \lambda_n) \right) \succeq 0 \]
- \(B = H + E\), where \(E = VD V^T \succeq 0\)

**Question:** What are the properties of the resulting search direction?

\[ p = -B^{-1}g = -V \bar{\Lambda}^{-1} V^T g = -\sum_{j=1}^{n} \frac{v_j^T g}{\lambda_j} v_j \]

Taking norms and using the orthogonality of \(V\), gives

\[ \|p\|^2 = \|B^{-1}g\|^2 = \sum_{j=1}^{n} \frac{(v_j^T g)^2}{\lambda_j^2} = \sum_{\lambda_j \geq \varepsilon} \frac{(v_j^T g)^2}{\lambda_j^2} + \sum_{\lambda_j < \varepsilon} \frac{(v_j^T g)^2}{\varepsilon^2} \]

Thus, we conclude that

\[ \|p\|_2 = O\left(\frac{1}{\varepsilon}\right) \quad \text{provided } v_j^T g \neq 0 \text{ for at least one } \lambda_j < \varepsilon \]

- the step \(p\) goes unbounded as \(\varepsilon \to 0\) provided \(v_j^T g \neq 0\) for at least one \(\lambda_j < \varepsilon\)
- any indefinite matrix will generally satisfy this property
- the next method that we discuss is better!
Method 2: small scale problems \( (n < 1000) \)

**Algorithm 2** Compute modified Newton matrix \( B \) from \( H \)

1: **input** \( H \)
2: Choose \( \beta > 1 \), the desired bound on the condition number of \( B \).
3: Compute the spectral decomposition \( H = V \Lambda V^T \).
4: if \( H = 0 \) then
5: Set \( \varepsilon = 1 \)
6: else
7: Set \( \varepsilon = \|H\|_2/\beta > 0 \)
8: end if
9: Compute \( \bar{\Lambda} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n) \) with \( \bar{\lambda}_j = \begin{cases} \lambda_j & \text{if } \lambda_j \geq \varepsilon \\ -\lambda_j & \text{if } \lambda_j \leq -\varepsilon \\ \varepsilon & \text{otherwise} \end{cases} \)
10: **return** \( B = V \bar{\Lambda} V^T > 0 \), which satisfies \( \text{cond}(B) \leq \beta \)

- replace small eigenvalues \( \lambda_i \) of \( H \) with \( \varepsilon \)
- replace “sufficiently negative” eigenvalues \( \lambda_i \) of \( H \) with \( -\lambda_i \)
- \( \bar{A} = \Lambda + D \), where \( D = \text{diag} \left( \max(0, -2\lambda_1, \varepsilon - \lambda_1), \ldots, \max(0, -2\lambda_n, \varepsilon - \lambda_n) \right) \geq 0 \)
- \( B = H + E \), where \( E = VD^T \geq 0 \)

Suppose that \( B = H + E \) is computed from Algorithm 2 so that

\[
E = B - H = V(\bar{A} - \Lambda)V^T
\]

and since \( V \) is orthogonal that

\[
\|E\|_2 = \|V(\bar{A} - \Lambda)V^T\|_2 = \|\bar{A} - \Lambda\|_2 = \max_j |\bar{\lambda}_j - \lambda_j|
\]

By definition

\[
\bar{\lambda}_j - \lambda_j = \begin{cases} 0 & \text{if } \lambda_j \geq \varepsilon \\ \varepsilon - \lambda_j & \text{if } -\varepsilon < \lambda_j < \varepsilon \\ -2\lambda_j & \text{if } \lambda_j \leq -\varepsilon \\ \end{cases}
\]

which implies that

\[
\|E\|_2 = \max_{1 \leq j \leq n} \{ 0, \varepsilon - \lambda_j, -2\lambda_j \}.
\]

However, since \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), we know that

\[
\|E\|_2 = \max \{ 0, \varepsilon - \lambda_n, -2\lambda_n \}
\]

- if \( \lambda_n \geq \varepsilon \), i.e., \( H \) is sufficiently positive definite, then \( E = 0 \) and \( B = H \)
- if \( \lambda_n < \varepsilon \), then \( B \neq H \) and it can be shown that \( \|E\|_2 \leq 2 \max(\varepsilon, |\lambda_n|) \)
- regardless, \( B \) is well-conditioned by construction
Example 2.2
Consider
\[ g = \begin{pmatrix} \frac{2}{4} \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \]

- The Newton direction is
  \[ p^N = -H^{-1}g = \begin{pmatrix} \frac{-2}{2} \\ \frac{2}{2} \end{pmatrix} \text{ so that } g^Tp^N = 4 > 0 \text{ (ascent direction)} \]
and \( p^N \) is a saddle point of the quadratic model \( g^T p + \frac{1}{2} p^T Hp \) since \( H \) is indefinite.

- Algorithm 1 (Method 1) generates
  \[ B = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \]
  so that
  \[ p = -B^{-1}g = \begin{pmatrix} \frac{-2}{2} \\ \frac{-2}{2} \end{pmatrix} \text{ and } g^T p = -4 - \frac{16}{\varepsilon} < 0 \text{ (descent direction)} \]

- Algorithm 2 (Method 2) generates
  \[ B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \]
  so that
  \[ p = -B^{-1}g = \begin{pmatrix} \frac{-2}{2} \\ \frac{-2}{2} \end{pmatrix} \text{ and } g^T p = -12 < 0 \text{ (descent direction)} \]
**Question:** What is the geometric interpretation?

**Answer:** Let the spectral decomposition of $H$ be

$$H = VAV^T$$

and assume that $\lambda_j \neq 0$ for all $j$. Partition the eigenvector and eigenvalue matrices as

$$A = \begin{pmatrix} A_+ & \{ \lambda_j > 0 \} \\ A_- & \{ \lambda_j < 0 \} \end{pmatrix}$$

and $V = \begin{pmatrix} V_+ & V_- \end{pmatrix}$

So that $H = (V_+ V_-) \begin{pmatrix} A_+ & \{ \lambda_j > 0 \} \\ A_- & \{ \lambda_j < 0 \} \end{pmatrix} \begin{pmatrix} V_+^T \\ V_-^T \end{pmatrix}$ and the Newton direction is

$$p^N = -VA^{-1}V_g = -(V_+ V_-) \begin{pmatrix} A_+^{-1} & \{ \lambda_j > 0 \} \\ A_-^{-1} & \{ \lambda_j < 0 \} \end{pmatrix} \begin{pmatrix} V_+^T_g \\ V_-^T_g \end{pmatrix}$$

$$= -V_+ A_+^{-1}V_+^T_g - V_- A_-^{-1}V_-^T_g$$

$$= \frac{-V_+ A_+^{-1}V_+^T_g}{\lambda_+} - \frac{V_- A_-^{-1}V_-^T_g}{\lambda_-}$$

where

$$p_+^N = -V_+ A_+^{-1}V_+^T_g \quad \text{and} \quad p_-^N = -V_- A_-^{-1}V_-^T_g$$

**Result**

Given $m_0^Q(p) = g^T p + \frac{1}{2} p^T Hp$ with $H$ nonsingular, the Newton direction

$$p^N = p_+^N + p_-^N = -V_+ A_+^{-1}V_+^T_g - V_- A_-^{-1}V_-^T_g$$

is such that

- $p_+^N$ minimizes $m_0^Q(p)$ on the subspace spanned by $V_+$.
- $p_-^N$ maximizes $m_0^Q(p)$ on the subspace spanned by $V_-$. 

Notes
Proof:

- The function $\psi(y) \equiv m_Q^k(V, y)$ may be written as
  
  $\psi(y) = g^T V y + \frac{1}{2} y^T H V y$

  and has a stationary point $y^*$ given by
  
  $y^* = -(V^T H V)^{-1} V^T y$

  Since
  
  $\nabla^2 \psi(y) = V^T H V = V^T A V^T = A \succ 0$

  we know that $y^*$ is, in fact, the unique minimizer of $\psi(y)$ and

  $V_y = -V A^{-1} V^T y \equiv \rho_N$

- For the second part
  
  $y^* = -(V^T H V_y)^{-1} V^T y = -A^{-1} V^T y$

  so that

  $V_y = -V A^{-1} V^T y \equiv \rho_N$

  Since $A \prec 0$, we know that $\rho_N$ maximizes $m_Q^k(p)$ on the space spanned by the columns of $V_y$.

Some observations

- $\rho_N$ is a descent direction for $f$:
  
  $g^T \rho_N = -g^T V A^{-1} V^T y = -y^T A^{-1} y \leq 0$

  where $y \equiv V^T y$. The inequality is strict if $y \neq 0$, i.e., $g$ is not orthogonal to the column space of $V_y$

- $\rho_N$ is an ascent direction for $f$:

  $g^T \rho_N = -g^T V A^{-1} V^T y = -y^T A^{-1} y \geq 0$

  where $y \equiv V^T y$. The inequality is strict if $y \neq 0$, i.e., $g$ is not orthogonal to the column space of $V_y$

- $\rho_N = \rho_N^+ + \rho_N^-$ may or may not be a descent direction!

- This is the geometry of the Newton direction. What about the modified Newton direction?
Question: what is the geometry of the modified-Newton step?

Answer: Partition the matrices in the spectral decomposition $H = VAV^T$ such that

$$V = [V_+ \ V_\varepsilon \ V_-] \quad \text{and} \quad \Lambda = \begin{pmatrix} \Lambda_+ & \Lambda_\varepsilon \\ \Lambda_\varepsilon & \Lambda_- \end{pmatrix}$$

where

$$\Lambda_+ = \{ \lambda_i : \lambda_i \geq \varepsilon \} \quad \Lambda_\varepsilon = \{ \lambda_i : |\lambda_i| < \varepsilon \} \quad \Lambda_- = \{ \lambda_i : \lambda_i \leq -\varepsilon \}$$

Theorem 2.3 (Properties of the modified-Newton direction)

Suppose that the positive-definite matrix $B_k$ is computed from Algorithm 2 with input $H_k$ and value $\varepsilon > 0$. If $g_k \neq 0$ and $p_k$ is the unique solution to $B_k p = -g_k$, then $p_k$ may be written as $p_k = p_+ + p_\varepsilon + p_-$, where

(i) $p_+$ is a direction of positive curvature that minimizes $m_k^0(p)$ in the space spanned by the columns of $V_+$;

(ii) $-(p_-)$ is a direction of negative curvature that maximizes $m_k^0(p)$ in the space spanned by the columns of $V_-; \text{ and}$

(iii) $p_\varepsilon$ is a multiple of the steepest-descent direction in the space spanned by the columns of $V_\varepsilon$ with $\|p_\varepsilon\|_2 = O(1/\varepsilon)$.

Comments:

- implies that singular Hessians $H_k$ may easily cause numerical difficulties
- the direction $p_-$ is the negative of $p_k^\infty$ associated with the Newton step

Main goals of quasi-Newton methods

- Maintain positive-definite “approximation” $B_k$ to $H_k$
- Instead of computing $B_{k+1}$ from scratch, e.g., from a spectral decomposition of $H_{k+1}$, want to compute $B_{k+1}$ as an update to $B_k$ using information at $x_{k+1}$
- inject “real” curvature information from $H(x_{k+1})$ into $B_{k+1}$
- Should be cheap to form and compute with $B_k$
- be applicable for medium-scale problems ($n \approx 1000 - 10,000$)
- Hope that fast convergence (superlinear) of the iterates $\{x_k\}$ computed using $B_k$ can be recovered
The optimization problem solved in the \((k + 1)\)st iterate will be

\[
\min_{p \in \mathbb{R}^n} m^0_{k+1}(p) \overset{\text{def}}{=} f_{k+1} + g_{k+1}^T p + \frac{1}{2} p^T B_{k+1} p
\]

How do we choose \(B_{k+1}\)?

How about to satisfy the following:

- the model should agree with \(f\) at \(x_{k+1}\)
- the gradient of the model should agree with the gradient of \(f\) at \(x_{k+1}\)
- the gradient of the model should agree with the gradient of \(f\) at \(x_k\) (previous point)

These are equivalent to

1. \(m^0_{k+1}(0) = f_{k+1}\) \(\checkmark\) already satisfied
2. \(\nabla m^0_{k+1}(0) = g_{k+1}\) \(\checkmark\) already satisfied
3. \(\nabla m^0_{k+1}(-\alpha_k p_k) = g_k \quad B_{k+1} (-\alpha_k p_k) + g_{k+1} = g_k \iff B_{k+1} (x_{k+1} - x_k) = g_{k+1} - g_k\)

Thus, we aim to cheaply compute a symmetric positive-definite matrix \(B_{k+1}\) that satisfies the so-called secant equation

\[
B_{k+1} s_k = y_k \quad \text{where} \quad s_k \overset{\text{def}}{=} x_{k+1} - x_k \quad \text{and} \quad y_k \overset{\text{def}}{=} g_{k+1} - g_k
\]

Observation: If the matrix \(B_{k+1}\) is positive definite, then the curvature condition

\[
s^T y_k > 0
\]

is satisfied.

- using the fact that \(B_{k+1} s_k = y_k\), we see that

\[
s^T y_k = s^T_k B_{k+1} s_k
\]

which must be positive if \(B_{k+1} \succ 0\)

- the previous relation explains why (1) is called a curvature condition

(1) will hold for any two points \(x_k\) and \(x_{k+1}\) if \(f\) is strictly convex

(1) does not always hold for any two points \(x_k\) and \(x_{k+1}\) if \(f\) is nonconvex

for nonconvex functions, we need to be careful how we choose \(\alpha_k\) that defines \(x_{k+1} = x_k + \alpha_k p_k\) (see Wolfe conditions for linesearch strategies)

Note: for the remainder, I will drop the subscript \(k\) from all terms.
The problem
Given a symmetric positive-definite matrix $B$, and vectors $s$ and $y$, find a symmetric positive-definite matrix $\bar{B}$ that satisfies the secant equation $\bar{B}s = y$

Want a cheap update, so let us first consider a rank-one update of the form $\bar{B} = B + uv^T$ for some vectors $u$ and $v$ that we seek.

Derivation:

**Case 1:** $Bs = y$
Choose $u = v = 0$ since $\bar{B}s = (B + uv^T)s = Bs + uv^Ts = y$

**Case 2:** $Bs \neq y$
If $u \neq 0$ or $v \neq 0$ then $\bar{B}s = (B + uv^T)s = Bs + uv^Ts = Bs \neq y$

So search for $u \neq 0$ and $v \neq 0$. Symmetry of $B$ and desired symmetry of $\bar{B}$ imply that $\bar{B} = \bar{B}^T$ implies $B + uv^T = B^T + vu^T$ which implies $uv^T = vu^T$.

But $uv^T = vu^T$, $u \neq 0$, and $v \neq 0$ imply that $v = \alpha u$ for some $\alpha \neq 0$ (exercise). Thus, we should search for an $\alpha \neq 0$ and $u \neq 0$ such that the secant condition is satisfied by $\bar{B} = B + \alpha uu^T$

From previous slide, we hope to find an $\alpha > 0$ and $u \neq 0$ so that $\bar{B} = B + \alpha uu^T$

satisfies $y = \bar{B}s = Bs + \alpha uu^Ts = Bs + \alpha (u^Ts)u$

which implies $\alpha (u^Ts)u = y - Bs \neq 0$.

This implies that $u = \beta (y - Bs)$ for some $\beta \neq 0$.

Plugging back in, we are searching for $\bar{B} = B + \alpha \beta^2 (y - Bs)(y - Bs)^T = B + \gamma (y - Bs)(y - Bs)^T$ for some $\gamma \neq 0$

that satisfies the secant equation $y = \bar{B}s = Bs + \gamma (y - Bs)^Ts(y - Bs)$

which implies $y - Bs = \gamma (y - Bs)^Ts(y - Bs)$

which means we can choose $\gamma = \frac{1}{(y - Bs)^Ts}$ provided $(y - Bs)^Ts \neq 0$. 

Definition 2.4 (SR1 quasi-Newton formula)

If \((y - Bs)^T s \neq 0\) for some symmetric matrix \(B\) and vectors \(s\) and \(y\), then the symmetric rank-1 (SR1) quasi-Newton formula for updating \(B\) is

\[
\bar{B} = B + \frac{(y - Bs)(y - Bs)^T}{(y - Bs)^T s}
\]

and satisfies

- \(\bar{B}\) is symmetric
- \(\bar{B}s = y\) (secant condition)

In practice, we choose some \(\kappa \in (0, 1)\) and then use

\[
\bar{B} = \begin{cases} 
B + \frac{(y - Bs)(y - Bs)^T}{(y - Bs)^T s} & \text{if } |s^T(y - Bs)| > \kappa \|s\|_2\|y - Bs\|_2 \\
B & \text{otherwise}
\end{cases}
\]

to avoid numerical error.

**Question**: Is \(\bar{B}\) positive definite if \(B\) is positive definite?

**Answer**: Sometimes,...but sometimes not!

Example 2.5 (SR1 update is not necessarily positive definite)

Consider the SR1 update with

\[
B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad s = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} -3 \\ 2 \end{pmatrix}
\]

The matrix \(B\) is positive definite with eigenvalues \(\approx \{0.38, 2.6\}\). Moreover,

\[
y^T s = (-3 \ 2) \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 1 > 0
\]

so that the necessary condition for \(\bar{B}\) to be positive definite holds. The SR1 update gives

\[
y - Bs = \begin{pmatrix} -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}
\]

\[(y - Bs)^T s = \begin{pmatrix} 0 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -4\]

\[
\bar{B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{(-4)} \begin{pmatrix} 0 \\ 4 \end{pmatrix} \begin{pmatrix} 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}
\]

The matrix \(\bar{B}\) is indefinite with eigenvalues \(\approx \{2.19, -3.19\}\).
Theorem 2.6 (convergence of SR1 update) 

Suppose that

- $f$ is $C^2$ on $\mathbb{R}^n$
- $\lim_{k \to \infty} x_k = x^*$ for some vector $x^*$
- $\nabla^2 f$ is bounded and Lipschitz continuous in a neighborhood of $x^*$
- the steps $s_k$ are uniformly linearly independent
- $|s_k^T (y_k - B_k s_k)| > \kappa \|s_k\|_2 \|y_k - B_k s_k\|_2$ for some $\kappa \in (0, 1)$

Then, the matrices generated by the SR1 formula satisfy

$$
\lim_{k \to \infty} B_k = \nabla^2 f(x^*)
$$

Comment: the SR1 updates may converge to an indefinite matrix

Some comments

- if $B$ is symmetric, then the SR1 update $\bar{B}$ (when it exists) is symmetric
- if $B$ is positive definite, then the SR1 update $\bar{B}$ (when it exists) is not necessarily positive definite
- linesearch methods require positive-definite matrices
- SR1 updating is not appropriate (in general) for linesearch methods
- SR1 updating is an attractive option for optimization methods that effectively utilize indefinite matrices (e.g., trust-region methods)
- to derive an updating strategy that produces positive-definite matrices, we have to look beyond rank-one updates. Would a rank-two update work?
For hereditary positive definiteness we must consider updates of at least rank two.

Result

$U$ is a symmetric matrix of rank two if and only if

$$U = \beta uu^T + \delta ww^T,$$

for some nonzero $\beta$ and $\delta$, with $u$ and $w$ linearly independent.

If

$$\bar{B} = B + U$$

for some rank-two matrix $U$ and $\bar{B}s = y$ for some vectors $s$ and $y$, then

$$\bar{B}s = (B + U)s = y,$$

and it follows that

$$y - Bs = Us = \beta(u^Ts)u + \delta(w^Ts)w.$$

Let us make the “obvious” choices of

$$u = Bs$$
and
$$w = y$$

which gives the rank-two update

$$U = \beta Bs(Bs)^T + \delta yy^T$$

and we now search for $\beta$ and $\delta$ so that $\bar{B}s = y$ is satisfied. Substituting, we have

$$y \overset{\text{want}}{=} \bar{B}s = (B + \beta Bs(Bs)^T + \delta yy^T)s$$
$$= Bs + \beta Bss^T Bs + \delta yy^T s$$
$$= Bs + \beta(s^T Bs)Bs + \delta (y^Ts)y.$$

Making the “simple” choice of

$$\beta = -\frac{1}{s^T Bs}$$
and
$$\delta = \frac{1}{y^Ts}$$

gives

$$\bar{B}s = Bs - \frac{1}{s^T Bs} (s^T Bs)Bs + \frac{1}{y^Ts} (y^Ts)y$$
$$= Bs - Bs + y = y$$

as desired. This gives the Broyden, Fletcher, Goldfarb, Shanno (BFGS) quasi-Newton update formula.
Broyden, Fletcher, Goldfarb, Shanno (BFGS) update

Given a symmetric positive-definite matrix $B$ and vectors $s$ and $y$ that satisfy $y^T s > 0$, the quasi-Newton BFGS update formula is given by

$$\bar{B} = B - \frac{1}{s^T B s} B s (B s)^T + \frac{1}{y^T s} y y^T$$

(2)

Result

If $B$ is positive definite and $y^T s > 0$, then

$$\bar{B} = B - \frac{1}{s^T B s} B s (B s)^T + \frac{1}{y^T s} y y^T$$

is positive definite and satisfies $\bar{B} s = y$.

Proof:

Homework assignment. (hint: see the next slide)

A second derivation of the BFGS formula

Let $M = B^{-1}$ for some given symmetric positive definite matrix $B$. Solve

$$\minimize_{\bar{M} \in \mathbb{R}^{n \times n}} \|\bar{M} - M\|_W \text{ subject to } \bar{M} = \bar{M}^T, \ \bar{M} y = s$$

for a “closest” symmetric matrix $\bar{M}$, where

$$\|A\|_W = \|W^{1/2} A W^{1/2}\|_F \quad \text{and} \quad \|C\|_F = \sum_{i=1}^n \sum_{j=1}^n c_{ij}^2$$

for any weighting matrix $W$ that satisfies $W y = s$. If we choose

$$W = G^{-1} \quad \text{and} \quad G = \int_0^1 [\nabla^2 f(x + \tau s)]^{-1} d\tau$$

then the solution is

$$\bar{M} = \left( I - \frac{s y^T}{y^T s} \right) M \left( I - \frac{y y^T}{y^T s} \right) + \frac{s y^T}{y^T s}$$

Computing the inverse $\bar{B} = \bar{M}^{-1}$ using the Sherman-Morrison-Woodbury formula gives

$$\bar{B} = B - \frac{1}{s^T B s} B s (B s)^T + \frac{1}{y^T s} y y^T$$

(BFGS formula again)

so that $\bar{B} \succ 0$ if and only if $\bar{M} \succ 0$. 

Notes
Recall: given the positive-definite matrix $B$ and vectors $s$ and $y$ satisfying $s^T y > 0$, the BFGS formula (2) may be written as

$$B = B - aa^T + bb^T$$

where

$$a = \frac{Bs}{(s^T Bs)^{1/2}} \quad \text{and} \quad b = \frac{y}{(y^T s)^{1/2}}$$

**Limited-memory BFGS update**

Suppose at iteration $k$ we have $m$ previous pairs $(s_i, y_i)$ for $i = k - m, \ldots, k - 1$. Then the limited-memory BFGS (L-BFGS) update with memory $m$ and positive-definite seed matrix $B_0^k$ may be written in the form

$$B_k = B_k^0 + \sum_{i=k-m}^{k-1} \left[ b_i b_i^T - a_i a_i^T \right]$$

for some vectors $a_i$ and $b_i$ that may be recovered via an unrolling formula (next slide).

- the positive-definite seed matrix $B_0^k$ is often chosen to be diagonal
- essentially perform $m$ BFGS updates to the seed matrix $B_0^k$
- A different seed matrix may be used for each iteration $k$
  - Added flexibility allows “better” seed matrices to be used as iterations proceed

**Algorithm 3 Unrolling the L-BFGS formula**

1. **input** $m \geq 1, \{(s_i, y_i)\}$ for $i = k - m, \ldots, k - 1$, and $B_0^k \succ 0$
2. **for** $i = k - m, \ldots, k - 1$ **do**
3. $b_i \leftarrow y_i / (y_i^T s_i)^{1/2}$
4. $a_i \leftarrow B_0^k s_i + \sum_{j=k-m}^{i-1} [(b_j^T s_i) b_j - (a_j^T s_i) a_j]
5. a_i \leftarrow a_i / (s_i^T a_i)^{1/2}$
6. **end for**

- typically $3 \leq m \leq 30$
- $b_i$ and $b_j^T s_i$ should be saved and reused from previous iterations (exceptions are $b_{k-m}$ and $b_{k-m}^T s_{k-m}$ because they depend on the most recent data obtained).
- with the previous savings in computation and assuming $B_0^k = I$, the total cost to get the $a_i, b_i$ vectors is approximately $\frac{3}{2}m^2 n$.
- cost of a matrix-vector product with $B_k$ requires $4mn$ multiplications.
- other so-called compact representations are slightly more efficient
The problem of interest

Given a symmetric positive-definite matrix $A$, solve the linear system

$$Ax = b$$

which is equivalent to finding the unique minimizer of

$$\min_{x \in \mathbb{R}^n} q(x) \overset{\text{def}}{=} \frac{1}{2} x^T Ax - b^T x$$

because $\nabla q(x) = Ax - b$ and $\nabla^2 q(x) = A \succ 0$ implies that

$$\nabla q(x) = Ax - b = 0 \iff Ax = b$$

We seek a method that cheaply computes approximate solutions to the system $Ax = b$, or equivalent approximately minimizes (3).

Do not want to compute a factorization of $A$, because it is assumed to be too expensive. We are interested in very large-scale problems.

We allow ourselves to compute matrix-vector products, i.e., $Av$ for any vector $v$.

Why do we care about this? We will see soon!

Notation: $r(x) \overset{\text{def}}{=} Ax - b$ and $r_k \overset{\text{def}}{=} Ax_k - b$.

Algorithm 4  Coordinate descent algorithm for minimizing $q$

1: **input** initial guess $x_0 \in \mathbb{R}^n$ and a symmetric positive-definite matrix $A \in \mathbb{R}^{n \times n}$
2: **loop**
3: for $i = 0, 1, \ldots, n - 1$ do
4: Compute $\alpha_i$ as the solution to
5: $$\min_{\alpha \in \mathbb{R}} q(x_i + \alpha e_{i+1})$$
6: $x_{i+1} \leftarrow x_i + \alpha_i e_{i+1}$
7: end for
8: $x_0 \leftarrow x_n$
9: **end loop**

- $e_j$ represents the $j$th coordinate vector, i.e.,

$$e_j = (0 \ 0 \ldots \ 1 \ 0 \ldots 0)^T \in \mathbb{R}^n$$

- Is it easy to solve (4)? Yes! (exercise)
- Does this work?
Fact: the axes are determined by the eigenvectors of $A$.

Observation: coordinate descent converges in no more than $n$ steps if the eigenvectors align with the coordinate directions, i.e., $V = I$ where $V$ is the eigenvector matrix of $A$.

Implication: Using the spectral decomposition of $A$

$$A = V\Lambda V^T = I\Lambda I^T = \Lambda$$

(a diagonal matrix)

Conclusion: Coordinate descent converges in $\leq n$ steps when $A$ is diagonal.

First thought: This is only good for diagonal matrices. That sucks!

Second thought: Can we use a transformation of variables so that the transformed Hessian is diagonal?

- Let $S$ be a square nonsingular matrix and define the transformed variables

$$y = S^{-1}x \iff Sy = x.$$  

Consider the transformed quadratic function $\tilde{q}$ defined as

$$\tilde{q}(y) \overset{\text{def}}{=} q(Sy) \equiv \frac{1}{2}y^T S^T AS y - (S^T b)^T y = \frac{1}{2}y^T \tilde{A} y - \tilde{b}^T y$$

so that a minimizer $x^*$ of $q$ satisfies $x^* = Sy^*$ where $y^*$ is a minimizer of $\tilde{q}$.

- If $\tilde{A} = S^T AS$ is diagonal, then applying the coordinate descent Algorithm 4 to $\tilde{q}$ would converge to $y^*$ in $\leq n$ steps.

- $A$ is diagonal if and only if

$$s_i^T A s_j = 0 \quad \text{for all } i \neq j,$$

where

$$S = (s_0 \ s_1 \ \ldots \ s_{n-1})$$

- Line 5 of Algorithm 4 (in the transformed variables) is

$$y_{i+1} = y_i + \alpha_i e_i$$

Multiplying by $S$ and using $Sy = x$ leads to

$$x_{i+1} = Sy_{i+1} = S(y_i + \alpha_i e_{i+1}) = Sy_i + \alpha_i S e_{i+1} = x_i + \alpha_i s_i$$

Notes

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A set of nonzero vectors \( \{s_0, s_1, \ldots, s_l\} \) is said to be conjugate with respect to the symmetric positive-definite matrix \( A \) (\( A \)-conjugate for short) if
\[
s_i^T A s_j = 0 \quad \text{for all} \quad i \neq j.
\]

We want to find \( A \)-conjugate vectors.

The eigenvectors of \( A \) are \( A \)-conjugate since the spectral decomposition gives
\[
A = V \Lambda V^T \iff V^T A V = \Lambda \quad \text{(diagonal matrix of eigenvalues)}
\]

Computing the spectral decomposition is too expensive for large scale problems.

We want to find \( A \)-conjugate vectors cheaply!

Let us look at a general algorithm.

**Algorithm 5** General \( A \)-conjugate direction algorithm

1. **input** symmetric positive-definite matrix \( A \in \mathbb{R}^{n \times n} \), and vectors \( x_0 \) and \( b \in \mathbb{R}^n \).
2. Set \( r_0 \leftarrow Ax_0 - b \) and \( k \leftarrow 0 \).
3. Choose direction \( s_k \).
4. **while** \( r_k \neq 0 \) **do**
5. Compute \( \alpha_k \leftarrow \arg \min_{\alpha \in \mathbb{R}} q(x_k + \alpha s_k) \)
6. Set \( x_{k+1} \leftarrow x_k + \alpha s_k \).
7. Set \( r_{k+1} \leftarrow A x_{k+1} - b \).
8. Pick new \( A \)-conjugate direction \( s_{k+1} \).
9. Set \( k \leftarrow k + 1 \).
10. **end while**

How do we compute \( \alpha_k \)?

How do we compute the \( A \)-conjugate directions efficiently?

**Theorem 2.8** (general \( A \)-conjugate algorithm converges)

For any \( x_0 \in \mathbb{R}^n \), the general \( A \)-conjugate direction Algorithm 5 computes the solution \( x^* \) of the system \( Ax = b \) in at most \( n \) steps.

**Proof:** Theorem 5.1 in Nocedal and Wright.
Exercise: show that the \( \alpha_k \) that solves the minimization problem on line 5 of Algorithm 5 satisfies

\[
\alpha_k = - \frac{s_k^T r_k}{s_k^T A s_k}.
\]

Before discussing how we compute the \( A \)-conjugate directions \( \{s_i\} \), let us look at some properties of them if we assume we already have them.

**Theorem 2.9 (Expanding subspace minimization)**

Let \( x_0 \in \mathbb{R}^n \) be any starting point and suppose that \( \{x_k\} \) is the sequence of iterates generated by the \( A \)-conjugate direction Algorithm 5 for a set of \( A \)-conjugate directions \( \{s_k\} \). Then,

\[
r_k^T s_i = 0 \quad \text{for all } i = 0, \ldots, k - 1
\]

and \( x_k \) is the unique minimizer of \( q(x) = \frac{1}{2} x^T A x - b^T x \) over the set

\[
\{ x : x = x_0 + \text{span}(s_0, s_1, \ldots, s_{k-1}) \}
\]

**Proof:** See Theorem 5.2 in Nocedal and Wright.

**Question:** If \( \{s_i\}_{i=0}^{k-1} \) are \( A \)-conjugate, how do we compute \( s_k \) to be \( A \)-conjugate?

**Answer:** Theorem 2.9 shows that \( r_k \) is orthogonal to the previously explored space. To keep it simple and hopefully cheap, let us try

\[
s_k = -r_k + \beta s_{k-1}
\]

(5)

If this is going to work, then the \( A \)-conjugacy condition and (5) requires that

\[
0 = s_k^T A s_k = s_{k-1}^T A (-r_k + \beta s_{k-1}) = s_{k-1}^T (-A r_k + \beta A s_{k-1}) = -s_{k-1}^T A r_k + \beta s_{k-1}^T A s_{k-1}
\]

and solving for \( \beta \) gives

\[
\beta = \frac{s_{k-1}^T A r_k}{s_{k-1}^T A s_{k-1}}.
\]

With this choice of \( \beta \), the vector \( s_k \) is \( A \)-conjugate!...but, how do we get started? Choosing \( s_0 = -r_0 \), i.e., a steepest descent direction, makes sense.

This gives us a complete algorithm: the linear CG algorithm!
**Algorithm 6** Preliminary linear CG algorithm

1: **input** symmetric positive-definite matrix $A \in \mathbb{R}^{n \times n}$, and vectors $x_0$ and $b \in \mathbb{R}^n$.
2: Set $r_0 \leftarrow Ax_0 - b$, $s_0 \leftarrow -r_0$, and $k \leftarrow 0$.
3: **while** $r_k \neq 0$ **do**
   4: Set $\alpha_k \leftarrow -(s_k^T r_k) / (s_k^T A s_k)$.
   5: Set $x_{k+1} \leftarrow x_k + \alpha_k s_k$.
   6: Set $r_{k+1} \leftarrow A x_{k+1} - b$.
   7: Set $\beta_{k+1} \leftarrow (s_{k+1}^T A s_{k+1}) / (s_k^T A s_k)$.
   8: Set $s_{k+1} \leftarrow -r_{k+1} + \beta_{k+1} s_k$.
   9: Set $k \leftarrow k + 1$.
10: **end while**

- This preliminary version requires two matrix-vector products per iteration.
- By using some more “tricks” and algebra, we can reduce this to a single matrix-vector multiplication per iteration.

**Algorithm 7** Linear CG algorithm

1: **input** symmetric positive-definite matrix $A \in \mathbb{R}^{n \times n}$, and vectors $x_0$ and $b \in \mathbb{R}^n$.
2: Set $r_0 \leftarrow Ax_0 - b$, $s_0 \leftarrow -r_0$, and $k \leftarrow 0$.
3: **while** $r_k \neq 0$ **do**
   4: Set $\alpha_k \leftarrow (r_k^T r_k) / (s_k^T A s_k)$.
   5: Set $x_{k+1} \leftarrow x_k + \alpha_k s_k$.
   6: Set $r_{k+1} \leftarrow r_k + \alpha_k A s_k$.
   7: Set $\beta_{k+1} \leftarrow (r_{k+1}^T r_{k+1}) / (r_k^T r_k)$.
   8: Set $s_{k+1} \leftarrow -r_{k+1} + \beta_{k+1} s_k$.
   9: Set $k \leftarrow k + 1$.
10: **end while**

- Only requires the single matrix-vector multiplication $A s_k$.
- More practical to use a stopping condition like
  \[
  ||r_k|| \leq 10^{-8} \max (1, ||r_0||_2)
  \]
- Required computation
  - 2 inner products
  - 3 vector sums
  - 1 matrix-vector multiplication
- Ideal for large (sparse) matrices $A$.
- Converges quickly if $\text{cond}(A)$ is small or the eigenvalues are “clustered”
  - Convergence can be accelerated by using preconditioning.
We want to use CG to compute an approximate solution $p$ to
\[
\min_{p \in \mathbb{R}^n} f + g^T p + \frac{1}{2} p^T H p
\]
- $H$ may be indefinite (even singular) and must ensure that $p$ is a descent direction

**Algorithm 8** Newton CG algorithm for computing descent directions

1. **input** symmetric matrix $H \in \mathbb{R}^{n \times n}$ and vector $g$
2. Set $p_0 = 0$, $r_0 \leftarrow -g$, $s_0 \leftarrow -g$, and $k \leftarrow 0$.
3. **while** $r_k \neq 0$ **do**
4.  **if** $s_k^T H s_k > 0$ **then**
5.      Set $\alpha_k \leftarrow (r_k^T r_k)/(s_k^T H s_k)$.
6.      **else**
7.        **if** $k = 0$, then return $p_k = -g$ else return $p_k$ **end if**
8.      **end if**
9.    Set $p_{k+1} \leftarrow p_k + \alpha_k s_k$.
10.   Set $r_{k+1} \leftarrow r_k + \alpha_k s_k$.
11.   Set $\beta_{k+1} \leftarrow (r_{k+1}^T r_{k+1})/(r_k^T r_k)$.
12.   Set $s_{k+1} \leftarrow -r_{k+1} + \beta_{k+1} s_k$.
13.   Set $k \leftarrow k + 1$.
14.  **end while**
15. **return** $p_k$

Made the replacements $x \leftarrow p$, $A \leftarrow H$, and $b \leftarrow -g$ in Algorithm 7.

**Theorem 2.10** (Newton CG algorithm computes descent directions.)

If $g \neq 0$ and Algorithm 8 terminates on iterate $K$, then every iterate $p_k$ of the algorithm satisfies $p_k^T g < 0$ for $k \leq K$.

**Proof:**

**Case 1:** Algorithm 8 terminates on iteration $K = 0$.
It is immediate that $g^T p_0 = -g^T g = -\|g\|^2 < 0$ since $g \neq 0$.

**Case 2:** Algorithm 8 terminates on iteration $K > 1$.
First observe from $p_0 = 0$, $s_0 = -g$, $r_0 = g$, $g \neq 0$, and lines 10 and 9 of Algorithm 8 that
\[
g^T p_1 = g^T (p_0 + \alpha_0 s_0) = \alpha_0 g^T s_0 = \frac{r_0^T r_0}{s_0^T H s_0} g^T s_0 = -\|g\|^2 g^T H g < 0.
\]

(6)
By unrolling the “p” update, using A-conjugacy of $\{s_i\}$, and definition of $r_k$, we get
\[
s_k^T r_k = s_k^T (H p_k + g) = s_k^T g + s_k^T H \sum_{j=0}^{k-1} \alpha_j r_j = s_k^T g.
\]
The previous line, lines 9 and 5 of Algorithm 8, and the fact that $s_k^T r_k = -\|r_k\|^2$ gives
\[
g^T p_{k+1} = g^T p_k + \alpha_k g^T s_k = g^T p_k + \alpha_k s_k^T r_k = g^T p_k + \frac{r_k^T r_k}{s_k^T H s_k} s_k^T r_k = g^T p_k - \frac{\|r_k\|^2}{s_k^T H s_k} < g^T p_k.
\]
Combining the previous inequality and (6) results in
\[
g^T p_{k+1} < g^T p_k < \cdots < g^T p_1 < -\|g\|^2 g^T H g < 0.
\]
Algorithm 9  Backtracking

1: input $x_k, p_k$
2: Choose $\alpha_{\text{init}} > 0$ and $\tau \in (0, 1)$
3: Set $\alpha^{(0)} = \alpha_{\text{init}}$ and $l = 0$
4: until $f(x_k + \alpha^{(l)} p_k) \leq f(x_k)$
5: Set $\alpha^{(l+1)} = \tau \alpha^{(l)}$
6: Set $l \leftarrow l + 1$
7: end until
8: return $\alpha_k = \alpha^{(l)}$

- typical choices (not always)
  - $\alpha_{\text{init}} = 1$
  - $\tau = 1/2$
- this prevents the step from getting too small
- does not prevent the step from being too long
- decrease in $f$ may not be proportional to the directional derivative
- need to tighten the requirement that
  $$f(x_k + \alpha^{(l)} p_k) \leq f(x_k)$$

The Armijo condition

Given a point $x_k$ and search direction $p_k$, we say that $\alpha_k$ satisfies the Armijo condition if

$$f(x_k + \alpha_k p_k) \leq f(x_k) + \eta \alpha_k g_k^T p_k$$  \hspace{1cm} (7)

for some $\eta \in (0, 1)$, e.g., $\eta = 0.1$ or even $\eta = 0.0001$

Purpose: ensure the decrease in $f$ is substantial relative to the directional derivative.
Algorithm 10 A backtracking-Armijo linesearch

1: \textbf{input} $x_k, p_k$
2: Choose $\alpha_{\text{init}} > 0$, $\eta \in (0, 1)$, and $\tau \in (0, 1)$
3: Set $\alpha^{(0)} = \alpha_{\text{init}}$ and $l = 0$
4: \textbf{until} $f(x_k + \alpha^{(l)} p_k) \leq f(x_k) + \eta \alpha^{(l)} g_k^T p_k$
5: Set $\alpha^{(l+1)} = \tau \alpha^{(l)}$
6: Set $l \leftarrow l + 1$
7: \textbf{end until}
8: \textbf{return} $\alpha_k = \alpha^{(l)}$

- Does this always terminate?
- Does it give us what we want?

Theorem 3.1 (Satisfying the Armijo condition)

Suppose that
- $f \in C^1$ and $g(x)$ is Lipschitz continuous with Lipschitz constant $\gamma(x)$
- $p$ is a descent direction at $x$

Then for any given $\eta \in (0, 1)$, the Armijo condition

$$f(x + \alpha p) \leq f(x) + \eta \alpha g(x)^T p$$

is satisfied for all $\alpha \in [0, \alpha_{\text{max}}(x)]$, where

$$\alpha_{\text{max}}(x) = \frac{2(\eta - 1)g(x)^T p}{\gamma(x)\|p\|_2^2} > 0$$

\begin{proof}
A Taylor's approximation and

$$\alpha \leq \frac{2(\eta - 1)g(x)^T p}{\gamma(x)\|p\|_2^2}$$

imply that

$$f(x + \alpha p) \leq f(x) + \alpha g(x)^T p + \frac{1}{2}\gamma(x)\alpha^2\|p\|_2^2$$

$$\leq f(x) + \alpha g(x)^T p + \alpha(\eta - 1)g(x)^T p$$

$$= f(x) + \alpha\eta g(x)^T p$$
\end{proof}
Theorem 3.2 (Bound on $\alpha_k$ when using an Armijo backtracking linesearch)

Suppose that
- $f \in C^1$ and $g(x)$ is Lipschitz continuous with Lipschitz constant $\gamma_k$ at $x_k$
- $p_k$ is a descent direction at $x_k$

Then for any chosen $\eta \in (0, 1)$ the stepsizes $\alpha_k$ generated by the backtracking-Armijo linesearch terminates with

$$
\alpha_k \geq \min \left( \alpha_{\text{init}}, \frac{2\tau(\eta - 1)g^T p_k}{\gamma_k || p_k ||^2} \right) \equiv \min \left( \alpha_{\text{init}}, \tau \alpha_{\text{max}}(x_k) \right)
$$

where $\tau \in (0, 1)$ is the backtracking parameter.

Proof:
Theorem 3.1 implies that the backtracking will terminate as soon as $\alpha^{(l)} \leq \alpha_{\text{max}}(x_k)$.
Consider the following two cases:
- $\alpha_{\text{init}}$ satisfies the Armijo condition: it is then clear that $\alpha_k = \alpha_{\text{init}}$.
- $\alpha_{\text{init}}$ does not satisfy the Armijo condition: thus, there must be a last linesearch iterate (the $l$-th) for which

$$
\alpha^{(l)} > \alpha_{\text{max}}(x_k) \implies \alpha_k = \alpha^{(l+1)} = \tau \alpha^{(l)} > \tau \alpha_{\text{max}}(x_k)
$$

Combining the two cases gives the required result.

Algorithm 11  Generic backtracking-Armijo linesearch method

1: input initial guess $x_0$
2: Set $k = 0$
3: loop
4: Find a descent direction $p_k$ at $x_k$.
5: Compute the stepsize $\alpha_k$ using the backtracking-Armijo linesearch Algorithm 10.
6: Set $x_{k+1} = x_k + \alpha_k p_k$
7: Set $k \leftarrow k + 1$
8: end loop
9: return approximate solution $x_k$

- should include a sensible relative stopping tolerance such as $|| \nabla f(x_k) || \leq 10^{-8} \max \left(1, || \nabla f(x_0) || \right)$
- should also terminate if a maximum number of allowed iterations is reached
- should also terminate if a maximum time limit is reached
Theorem 3.3 (Global convergence of generic backtracking-Armijo linesearch)

Suppose that
- \( f \in C^1 \)
- \( g \) is Lipschitz continuous on \( \mathbb{R}^n \) with Lipschitz constant \( \gamma \)

Then, the iterates generated by the generic backtracking-Armijo linesearch method Algorithm 11 must satisfy one of the following conditions:

1. **finite termination**, i.e., there exists a positive integer \( k \) such that
   \[ g_k = 0 \text{ for some } k \geq 0 \]

2. **objective is unbounded below**, i.e.,
   \[ \lim_{k \to \infty} f_k = -\infty \]

3. **an angle condition** between \( g_k \) and \( p_k \) exists, i.e.,
   \[ \lim_{k \to \infty} \min \left( |p_k^T g_k|, |p_k^T g_k|/\|p_k\|_2 \right) = 0 \]

**Proof:**
Suppose that outcomes 1 and 2 do not happen, i.e., \( g_k \neq 0 \) for all \( k \geq 0 \) and

\[ \lim_{k \to \infty} f_k > -\infty \tag{8} \]

The Armijo condition implies that

\[ f_{k+1} - f_k \leq \eta \alpha_k p_k^T g_k \]

for all \( k \). Summing over the first \( j \) iterations yields

\[ f_{j+1} - f_0 \leq \eta \sum_{k=0}^{j} \alpha_k p_k^T g_k \tag{9} \]

Taking limits of the previous inequality as \( j \to \infty \) and using (8) implies that the LHS is bounded below and, therefore, the RHS is also bounded below. Since the sum is composed of all negative terms, we deduce that

\[ \lim_{k \to \infty} \alpha_k |p_k^T g_k| = 0 \tag{10} \]
Define the sets 
\[ K_1 \triangleq \{ k : \alpha_{\text{init}} > \frac{2\tau(\eta - 1)g_k^T p_k}{\gamma\|p_k\|^2} \} \quad \text{and} \quad K_2 \triangleq \{ k : k \not\in K_1 \} \]
where \( \gamma \) is the uniform Lipschitz constant, \( \eta \in (0, 1) \) is the Armijo condition constant, and \( \tau \in (0, 1) \) is the backtracking parameter. Either \( K_1 \) or \( K_2 \) is an infinite set (subsequence).

- **Case 1:** \( K_1 \) is a subsequence. For \( k \in K_1 \)

\[
\alpha_k \geq \frac{2\tau(\eta - 1)g_k^T p_k}{\gamma\|p_k\|^2}
\]

which implies
\[
\alpha_k p_k^T g_k \leq \frac{2\tau(\eta - 1)}{\gamma} \left( \frac{g_k^T p_k}{\|p_k\|^2} \right)^2 < 0
\]

which combined with (10) implies
\[
\lim_{k \in K_1 \to \infty} \frac{p_k^T g_k}{\|p_k\|^2} = 0. \tag{11}
\]

- **Case 2:** \( K_2 \) is a subsequence. For \( k \in K_2 \)

\[
\alpha_k \geq \alpha_{\text{init}}
\]

which combined with (10) implies that
\[
\lim_{k \in K_2 \to \infty} p_k^T g_k = 0. \tag{12}
\]

Combining (11) and (12) gives the required result.

Some observations
- the Armijo condition prevents too “long” of steps
- typically, the Armijo condition is satisfied for very “large” intervals
- the Armijo condition can be satisfied by steps that are not even close to a minimizer of \( \phi(x) := f(x_k + \alpha p_k) \)
Wolfe conditions

Given the current iterate $x_k$, search direction $p_k$, and constants $0 < c_1 < c_2 < 1$, we say that the step length $\alpha_k$ satisfies the Wolfe conditions if

$$f(x_k + \alpha_k p_k) \leq f(x_k) + c_1 \alpha_k \nabla f(x_k)^T p_k$$ \hspace{1cm} (13a)

$$\nabla f(x_k + \alpha_k p_k)^T p_k \geq c_2 \nabla f(x_k)^T p_k$$ \hspace{1cm} (13b)

- condition (13a) is the Armijo condition (ensures "sufficient" decrease)
- condition (13b) is a directional derivative condition (prevents too short of steps)

Algorithm 12: Generic Wolfe linesearch method

1. **input** initial guess $x_0$
2. Set $k \leftarrow 0$
3. **loop**
   4. Find a descent direction $p_k$ at $x_k$.
   5. Compute a stepsize $\alpha_k$ that satisfies the Wolfe conditions (13).
   6. Set $x_{k+1} = x_k + \alpha_k p_k$
   7. Set $k \leftarrow k + 1$
4. **end loop**
5. **return** approximate solution $x_k$

- should include a sensible relative stopping tolerance such as
  $$\|\nabla f(x_k)\| \leq 10^{-8} \max(1, \|\nabla f(x_0)\|)$$

- should also terminate if a maximum number of allowed iterations is reached
- should also terminate if a maximum time limit is reached

Notes
Theorem 3.4 (Convergence of generic Wolfe linesearch (Zoutendijk))

Assume that
- \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is bounded below on \( C^1 \) on \( \mathbb{R}^n \)
- \( \nabla f(x) \) is Lipschitz continuous on \( \mathbb{R}^n \) with constant \( L \).

It follows that the iterates generated from the generic Wolfe linesearch method satisfy

\[
\sum_{k \geq 0} \cos^2(\theta_k) \| g_k \|^2 < \infty \tag{14}
\]

where

\[
\cos(\theta_k) \overset{\text{def}}{=} \frac{-g_k^T p_k}{\| g_k \| \| p_k \|}. \tag{15}
\]

Comments
- definition (15) measures the angle between the gradient \( g_k \) and search direction \( p_k \)
  - \( \cos(\theta) \approx 0 \) implies \( g_k \) and \( p_k \) are nearly orthogonal
  - \( \cos(\theta) \approx 1 \) implies \( g_k \) and \( p_k \) are nearly parallel
- condition (14) ensures that the gradient converges to zero provided \( g_k \) and \( p_k \) stay away from being arbitrarily close to orthogonal

Proof: See Theorem 3.2 in Nocedal and Wright.

Some comments
- The Armijo linesearch
  - only requires function evaluations
  - may easily accept steps that are far from a univariate minimizer of \( \phi(\alpha) = f(x_k + \alpha p_k) \)
- The Wolfe linesearch
  - requires function and gradient evaluations
  - typically produces steps that are closer to the univariate minimizer of \( \phi \)
  - may still allow the acceptance of steps that are far from a univariate minimizer of \( \phi \)
  - "ideal" when search directions are computed from linear systems based on the BFGS quasi-Newton updating formula

Note: we have not yet said how to compute steps that satisfy the Wolfe conditions. We have not even shown that such steps exist! (coming soon)
**Strong Wolfe conditions**

Given the current iterate \( x_k \), search direction \( p_k \), and constants \( 0 < c_1 < c_2 < 1 \), we say that the step length \( \alpha_k \) satisfies the **Strong Wolfe conditions** if

\[
\begin{align*}
 f(x_k + \alpha p_k) &\leq f(x_k) + c_1 \alpha_k \nabla f(x_k)^T p_k \\
 |\nabla f(x_k + \alpha p_k)^T p_k| &\leq c_2 |\nabla f(x_k)^T p_k| 
\end{align*}
\]  

\( (16a) \quad (16b) \)

**Theorem 3.5**

Suppose that \( f \in \mathbb{R}^n \rightarrow \mathbb{R} \) is in \( C^1 \) and \( p_k \) is a descent direction at \( x_k \), and assume that \( f \) is bounded below along the ray \( \{x_k + \alpha p_k : \alpha > 0\} \). It follows that if \( 0 < c_1 < c_2 < 1 \), then there exist nontrivial intervals of step lengths that satisfy the strong Wolfe conditions \( (16) \).

**Proof:**

The function \( \phi(\alpha) := f(x_k + \alpha p_k) \) is bounded below over \( \alpha > 0 \) by assumption and must, therefore, intersect the graph of \( l(\alpha) := f(x_k) + \alpha c_1 \nabla f(x_k)^T p_k \) at least once since \( c_1 \in (0, 1) \); let \( \alpha_{min} > 0 \) be the smallest such intersecting value so that

\[
f(x_k + \alpha_{min} p_k) = f(x_k) + \alpha_{min} c_1 \nabla f(x_k)^T p_k.
\]  

\( (17) \)

It is clear that \( (13a)/(16a) \) hold for all \( \alpha \in (0, \alpha_{min}] \). The Mean Value Theorem now implies

\[
f(x_k + \alpha_{min} p_k) - f(x_k) = \alpha_{min} \nabla f(x_k + \alpha_{min} p_k)^T p_k \quad \text{for some } \alpha_M \in (0, \alpha_{min}).
\]

Combining the last equality with \( (17) \) and the facts that \( c_1 < c_2 \) and \( \nabla f(x_k)^T p_k < 0 \) yield

\[
\nabla f(x_k + \alpha_{min} p_k)^T p_k = c_1 \nabla f(x_k)^T p_k > c_2 \nabla f(x_k)^T p_k.
\]  

\( (18) \)

Thus, \( \alpha_{min} \) satisfies \( (13a)/(16a) \) and \( (13b) \) with a strict inequality so we may use the smoothness assumption to deduce that there is an interval around \( \alpha_{min} \) for which \( (13) \) and \( (16a) \) hold. Finally, since the left-hand-side of \( (18) \) is negative, we also know that \( (16b) \) holds, which completes the proof.
Algorithm 13 A bisection type (weak) Wolfe linesearch

1: input $x_k, p_k$, $0 < c_1 < c_2 < 1$
2: Choose $\alpha = 0, t = 1$, and $\beta = \infty$
3: while $(f(x_k + tp_k) > f(x_k) + c_1 \cdot t \cdot (g_k^T p_k)$ OR $\nabla f(x_k + tp_k)^T p_k < c_2 (g_k^T p_k)$) do
4: if $f(x_k + tp_k) > f(x_k) + c_1 \cdot t \cdot (g_k^T p_k)$ then
5: Set $\beta := t$
6: Set $t := \frac{\alpha + \beta}{2}$
7: else
8: $\alpha := t$
9: if $\beta = \infty$ then
10: $t := 2\alpha$
11: else
12: $t := \frac{\alpha + \beta}{2}$
13: end if
14: end if
15: end while
16: return $t$

See Section 3.5 in Nocedal-Wright for more discussions; in particular, a procedure for the strong Wolfe conditions.

Some comments

- The Armijo linesearch
  - only requires function evaluations
  - computationally cheapest to implement
  - may easily accept steps that are far from a univariate minimizer of $\phi(\alpha) = f(x_k + \alpha p_k)$

- The Wolfe linesearch
  - requires function and gradient evaluations
  - typically produces steps that are closer to the univariate minimizer of $\phi$
  - may still allow the acceptance of steps that are far from a univariate minimizer of $\phi$
  - computationally more expensive to implement compared to Armijo

- The strong Wolfe linesearch
  - requires function and gradient evaluations
  - typically produces steps that are closer to the univariate minimizer of $\phi$
  - only approximate stationary points of the univariate function $\phi$ satisfy these conditions.
  - computationally most expensive to implement compared to Armijo

Why use (strong) Wolfe conditions?

- sometimes more accurate line search lead to fewer outer (major) iterations of the linesearch method.
- beneficial in the context of quasi-Newton methods for maintaining positive-definite matrix approximations.
Theorem 3.6 (Wolfe conditions guarantee the BFGS update is well-defined)

Suppose that $B_k \succ 0$ and that $p_k$ is a descent direction for $f$ at $x_k$. If a (strong) Wolfe linesearch is used to compute $\alpha_k$ such that $x_{k+1} = x_k + \alpha_k p_k$, then

$$y_k^T s_k > 0$$

where $s_k = x_{k+1} - x_k = \alpha_k p_k$ and $y_k = g_{k+1} - g_k$.

Proof:

It follows from (13b) that

$$\nabla f(x_k + \alpha_k p_k)^T p_k \geq c_2 \nabla f(x_k)^T p_k.$$  

Multiplying both sides by $\alpha_k$, we get

$$\nabla f(x_k + \alpha_k p_k)^T p_k \geq \alpha_k \nabla f(x_k)^T p_k.$$  

Subtracting $g_k^T s_k$ from both sides gives

$$\nabla f(x_k + \alpha_k p_k)^T p_k - g_k^T s_k \geq \alpha_k \nabla f(x_k)^T p_k - g_k^T s_k.$$  

so that

$$y_k^T s_k \geq (c_2 - 1) g_k^T s_k = \alpha_k (c_2 - 1) g_k^T p_k > 0$$

since $c_2 \in (0, 1)$ and $p_k$ is a descent direction for $f$ at $x_k$.  □

Algorithm 14  Steepest descent backtracking-Armijo linesearch method

1: Input: initial guess $x_0$
2: Set $k = 0$
3: until $\|\nabla f(x_k)\| \leq 10^{-8}$ max (1, $\|\nabla f(x_0)\|$)
4: Set $p_k = -\nabla f(x_k)$ as the search direction.
5: Compute the stepsize $\alpha_k$ using the backtracking-Armijo linesearch Algorithm 10.
6: Set $x_{k+1} = x_k + \alpha_k p_k$
7: Set $k \leftarrow k + 1$
8: end until
9: return approximate solution $x_k$

• should also terminate if a maximum number of allowed iterations is reached
• should also terminate if a maximum time limit is reached

Notes
Theorem 4.1 (Global convergence for steepest descent)

If \( f \in C^1 \) and \( g \) is Lipschitz continuous on \( \mathbb{R}^n \), then the iterates generated by the steepest-descent backtracking-Armijo line search Algorithm 14 satisfies one of the following conditions:

1. **finite termination**, i.e., there exists some finite integer \( k \geq 0 \) such that \( g_k = 0 \)
2. **objective is unbounded below**, i.e.,
   \[
   \lim_{k \to \infty} f_k = -\infty
   \]
3. **convergence of gradients**, i.e.,
   \[
   \lim_{k \to \infty} g_k = 0
   \]

**Proof:**

Since \( p_k = -g_k \), if outcome 1 above does not happen, then we may conclude that
\[
\|g_k\| = \|p_k\| \neq 0 \quad \text{for all } k \geq 0
\]
Additionally, if outcome 2 above does not occur, then Theorem 3.3 ensures that
\[
0 = \lim_{k \to \infty} \left\{ \min \left( \|p_k^T g_k\|, \|p_k^T g_k\|/\|p_k\|_2 \right) \right\} = \lim_{k \to \infty} \left\{ \|g_k\|_2 \min (\|g_k\|_2, 1) \right\}
\]
and thus \( \lim_{k \to \infty} g_k = 0 \). \( \blacksquare \)

Some general comments on the steepest descent backtracking-Armijo algorithm

- archtypical globally convergent method
- many other methods resort to steepest descent when things “go wrong”
- not scale invariant
- convergence is usually slow! (linear local convergence, possibly with constant close to 1; for global convergence only sublinear rate \( O((\frac{1}{\epsilon})^2) \) to get to a “stationary” point, i.e., \( \|g_k\| \leq \epsilon \))
- in practice, often does not converge at all
- slow convergence results because the computation of \( p_k \) does not use any curvature information
- call any method that uses the steepest descent direction a “method of steepest descent”
- each iteration is relatively cheap, but you may need many of them!
Rate of convergence

**Theorem 4.2 (Global convergence rate for steepest-descent)**

Suppose \( f \in C^1 \) and \( \nabla f \) is Lipschitz continuous on \( \mathbb{R}^n \). Further assume that \( f(x) \) is bounded from below on \( \mathbb{R}^n \). Let \( \{x_k\} \) be the iterates generated by the steepest-descent backtracking-Armijo linesearch Algorithm 14. Then there exists a constant \( M \) such that for all \( T \geq 1 \)

\[
\min_{k=0, \ldots, T} \| \nabla f(x_k) \| \leq \frac{M}{\sqrt{T} + 1}.
\]

Consequently, for any \( \epsilon > 0 \), within \( \lceil \left( \frac{M}{\epsilon} \right)^2 \rceil \) steps, we will see an iterate where the gradient has norm at most \( \epsilon \). In other words, we reach an “\( \epsilon \)-stationary” point in \( O((\frac{1}{\epsilon})^2) \) steps.

**REMARK:** Same thing holds for constant step size for an appropriately chosen constant. See HW.

**Proof:**

Recall (9) in the proof of Theorem 3.3:

\[
\sum_{k=0}^{T} \alpha_k |g_k^T p_k| \leq \frac{f_0 - f_{k+1}}{\eta} \leq M',
\]

since \( f \) is bounded from below.

Rate of convergence

Thus,

\[
\sum_{k : \tau \alpha_{\max} < \alpha_{\init}} \frac{2\tau(1 - \eta) |g_k^T p_k|^2}{\gamma |p_k|^2} + \sum_{k : \tau \alpha_{\max} \geq \alpha_{\init}} \alpha_{\init} |g_k^T p_k| \leq M'.
\]

Thus,

\[
\sum_{k : \tau \alpha_{\max} < \alpha_{\init}} \frac{|g_k^T p_k|^2}{|p_k|^2} + \sum_{k : \tau \alpha_{\max} \geq \alpha_{\init}} |g_k^T p_k| \leq M' \frac{C}{C'}, \tag{19}
\]

where \( C = \min \left\{ \frac{2\tau(1 - \eta)}{\gamma}, \alpha_{\init} \right\} \).

Setting \( p_k = -g_k \) in the inequality above, we obtain that

\[
(T + 1) \min_{k=0, \ldots, T} \| g_k \|^2 \leq \sum_{k=0, \ldots, T} \| g_k \|^2 \leq M' \frac{C}{C'}.
\]

Therefore,

\[
\min_{k=0, \ldots, T} \| g_k \| \leq \frac{M}{\sqrt{T + 1}},
\]

for some constant \( M \).
Steepest descent example

Figure: Contours for the objective function $f(x,y) = 10(y-x^2)^2 + (x-1)^2$, and the iterates generated by the steepest-descent backtracking-Armijo linesearch Algorithm 14.

Rate of convergence

**Theorem 4.3 (Local convergence rate for steepest-descent)**

Suppose the following all hold:

- $f$ is twice continuously differentiable and $\nabla^2 f$ is Lipschitz continuous on $\mathbb{R}^n$ with Lipschitz constant $M$.
- $x^*$ is a local minimum of $f$ with positive definite Hessian $\nabla^2 f(x^*)$ with eigenvalues lower bounded by $\ell$ and upper bounded by $L$.
- The initial starting iterate $x_0$ is close enough to $x^*$; more precisely, $r_0 := \|x_0 - x^*\| < \bar{r} := \frac{2\ell}{M}$.

The steepest-descent with constant step size $\alpha = \frac{2}{L^2 + \ell}$ converges as follows:

$$\|x_k - x^*\| \leq \frac{r_0}{\bar{r} - r_0} \left( 1 - \frac{2\ell}{L + 3\ell} \right)^k.$$  

In other words, in $O(\log(\frac{\epsilon}{\alpha}))$ iterations, we are within a distance of $\epsilon$ from $x^*$.

**Proof:** See Theorem 1.2.4 in Nesterov’s book.

See also Theorem 3.4 in Nocedal-Wright for a similar result with exact line-search.
Algorithm 15 Modified-Newton backtracking-Armijo linesearch method

1: Input: initial guess $x_0$
2: Set $k = 0$
3: until $\|\nabla f(x_k)\| \leq 10^{-8} \max (1, \|\nabla f(x_0)\|)$
4: Compute positive-definite matrix $B_k$ from $H_k$ using Algorithm 2.
5: Compute the search direction $p_k$ as the solution to $B_k p_k = -g_k$.
6: Compute the stepsize $\alpha_k$ using the backtracking-Armijo linesearch Algorithm 10.
7: Set $x_{k+1} = x_k + \alpha_k p_k$
8: Set $k \leftarrow k + 1$
9: end until
10: return approximate solution $x_k$

- should also terminate if a maximum number of allowed iterations is reached
- should also terminate if a maximum time limit is reached

Theorem 4.4 (Global convergence of modified-Newton backtracking-Armijo)

Suppose that
- $f \in C^1$
- $g$ is Lipschitz continuous on $\mathbb{R}^n$
- the eigenvalues of $B_k$ are uniformly bounded away from zero and infinity

Then, the iterates generated by the modified-Newton backtracking-Armijo Algorithm 15 satisfy one of the following:

- finite termination, i.e., there exists a finite $k$ such that $g_k = 0$
- objective is unbounded below, i.e., $\lim_{k \to \infty} f_k = -\infty$
- convergence of gradients, i.e., $\lim_{k \to \infty} g_k = 0$

Proof:
Let $\lambda_{\min}(B_k)$ and $\lambda_{\max}(B_k)$ be the smallest and largest eigenvalues of $B_k$. By assumption, there are bounds $0 < \lambda_{\min} \leq \lambda_{\max}$ such that

$$\lambda_{\min} \leq \lambda_{\min}(B_k) \leq \frac{s^T B_k s}{\|s\|^2} \leq \lambda_{\max}(B_k) \leq \lambda_{\max}$$

for any nonzero $s$. 

Notes
From the previous slide, it follows that
\[
\lambda_{\text{max}}^{-1} \leq \lambda_{\text{max}}^{-1}(B_k) = \lambda_{\text{min}}(B_k^{-1}) \leq \frac{s^T B_k^{-1} s}{\|s\|^2} \leq \lambda_{\text{max}}(B_k^{-1}) = \lambda_{\text{min}}^{-1}(B_k) \leq \lambda_{\text{max}}^{-1}
\]  
(20)
for any nonzero vector \(s\). Using \(B_k p_k = -g_k\) and (20) we see that
\[
|p_k^T g_k| = |g_k^T B_k^{-1} g_k| \geq \lambda_{\text{min}}^{-1} \|g_k\|^2.
\]  
(21)
In addition, we may use (20) to see that
\[
\|p_k\|^2 = g_k^T B_k^{-1} g_k \leq \lambda_{\text{max}} \|g_k\|^2,
\]
which implies that
\[
\|p_k\|_2 \leq \lambda_{\text{max}}^{-1} \|g_k\|_2.
\]
It now follows from the previous inequality and (21) that
\[
\frac{|p_k^T g_k|}{\|p_k\|_2} \geq \frac{\lambda_{\text{min}} \|g_k\|_2}{\lambda_{\text{max}} \|g_k\|_2}.
\]  
(22)
Combining the previous inequality and (21) yields
\[
\min \left( |p_k^T g_k|, \frac{|p_k^T g_k|}{\|p_k\|_2} \right) \geq \frac{\|g_k\|_2}{\lambda_{\text{max}}} \min \left( \lambda_{\text{min}}, \|g_k\|_2 \right).
\]  
(23)
From Theorem 3.3 we know that
\[
\lim_{k \to \infty} \min \left( |p_k^T g_k|, \frac{|p_k^T g_k|}{\|p_k\|_2} \right) = 0.
\]
and when combined with (23) leads us to conclude that
\[
\lim_{k \to \infty} g_k = 0.
\]

Rate of convergence

**Theorem 4.5 (Global convergence rate for modified-Newton)**

Suppose \(f \in C^1\) and \(\nabla f\) is Lipschitz continuous on \(\mathbb{R}^n\). Further assume that \(f(x)\) is bounded from below on \(\mathbb{R}^n\). Let \(\{x_k\}\) be the iterates generated by the modified-Newton backtracking-Armijo linesearch Algorithm 15. Then there exists a constant \(M\) such that for all \(T \geq 1\)
\[
\min_{k=0, \ldots, T} \|\nabla f(x_k)\| \leq \frac{M}{\sqrt{T+1}}.
\]
Consequently, for any \(\epsilon > 0\), within \(\lceil (\frac{M}{\epsilon})^2 \rceil\) steps, we will see an iterate where the gradient has norm at most \(\epsilon\). In other words, we reach an “\(\epsilon\)-stationary” point in \(O((\frac{1}{\epsilon})^2)\) steps.

**Proof:** Almost identical to steepest descent proof (Theorem 4.2); use (21) and (22) in (19).

**Remark:** Same thing holds for constant step size for an appropriately chosen constant.
Rate of convergence

**Theorem 4.6 (Local quadratic convergence for modified-Newton)**

Suppose that
- $f \in C^2$
- $H$ is Lipschitz continuous on $\mathbb{R}^n$ with constant $\gamma$

Suppose that iterates are generated from the modified-Newton backtracking-Armijo Algorithm 15 such that
- $\alpha_{\text{init}} = 1$
- $\eta \in (0, \frac{1}{2})$
- $B_k = H_k$ whenever $H_k$ is positive definite

If the sequence of iterates $\{x_k\}$ has a limit point $x^*$ satisfying $H(x^*)$ is positive definite, then it follows that

(i) $\alpha_k = 1$ for all sufficiently large $k$,
(ii) the entire sequence $\{x_k\}$ converges to $x^*$, and
(iii) the rate of convergence is $Q$-quadratic, i.e., there is a constant $\kappa \geq 0$ and a natural number $k_0$ such that for all $k \geq k_0$

$$\|x_{k+1} - x^*\|_2 \leq \kappa \|x_k - x^*\|_2^2$$

**Proof:**

By assumption we know that there is some subsequence $\mathcal{K}$ such that

$$\lim_{k \in \mathcal{K}} x_k = x^* \quad \text{and} \quad H(x^*) \succ 0$$

Continuity implies that $H_k$ is positive definite for all $k \in \mathcal{K}$ sufficiently large so that

$$p_k^T H_k p_k \geq \frac{1}{2} \lambda_{\min}(H(x^*)) \|p_k\|_2^2$$

for all $k_0 \leq k \in \mathcal{K}$ (24)

for some $k_0$, where $\lambda_{\min}(H(x^*))$ is the smallest eigenvalue of $H(x^*)$. This fact and since $H_k p_k = -g_k$ for $k_0 \leq k \in \mathcal{K}$ implies that

$$|p_k^T g_k| = |p_k^T g_k| = p_k^T H_k p_k \geq \frac{1}{2} \lambda_{\min}(H(x^*)) \|p_k\|_2^2$$

for all $k_0 \leq k \in \mathcal{K}$. (25)

This also implies

$$\frac{|p_k^T g_k|}{\|p_k\|_2} \geq \frac{1}{2} \frac{\lambda_{\min}(H(x^*))}{\|p_k\|_2}.$$ 

(26)

We now deduce from Theorem 3.3, (25), and (26) that

$$\lim_{k \in \mathcal{K} \to \infty} p_k = 0.$$
Mean Value Theorem implies that there exists $z_k$ between $x_k$ and $x_k + p_k$ such that

$$f(x_k + p_k) = f_k + p_k^T g_k + \frac{1}{2} p_k^T H(z_k) p_k.$$  

Lipschitz continuity of $H(x)$ and the fact that $H p_k = -g_k$ implies

$$f(x_k + p_k) - f_k - \frac{1}{2} p_k^T g_k = \frac{1}{2} (p_k^T g_k + p_k^T H(z_k) p_k)$$
$$= \frac{1}{2} p_k^T (H(z_k) - H_k) p_k$$
$$\leq \frac{1}{2} \|H(z_k) - H_k\|_2 \|p_k\|_2^2 \leq \frac{1}{2} \gamma \|p_k\|_2^2. \tag{27}$$

Since $\eta \in (0, 1/2)$ and $p_k \rightarrow 0$, we may pick $k$ sufficiently large so that

$$2\gamma \|p_k\|_2 \leq \lambda_{\text{min}}(H(x^*)) (1 - 2\eta). \tag{28}$$

Combining (27), (28), and (25) shows that

$$f(x_k + p_k) - f_k \leq \frac{1}{2} p_k^T g_k + \frac{1}{2} \gamma \|p_k\|_2^2$$
$$\leq \frac{1}{2} p_k^T g_k + \frac{1}{2} \lambda_{\text{min}}(H(x^*)) (1 - 2\eta) \|p_k\|_2^2$$
$$\leq \frac{3}{2} p_k^T g_k - \frac{1}{2} (1 - 2\eta) p_k^T g_k$$
$$= \eta p_k^T g_k$$

so that the unit step $x_k + p_k$ satisfies the Armijo condition for all $k \in \mathcal{K}$ sufficiently large.

Note that (24) implies that

$$\|H_k^{-1}\|_2 \leq 2/\lambda_{\text{min}}(H(x^*)) \quad \text{for all } k_0 \leq k \in \mathcal{K}. \tag{29}$$

The iteration gives

$$x_{k+1} - x^* = x_k - x^* - H_k^{-1} g_k = x_k - x^* - H_k^{-1} (g_k - g(x^*))$$
$$= H_k^{-1} [g(x^*) - g_k - H_k (x^* - x_k)].$$

But a Taylor Approximation provides the estimate

$$\|g(x^*) - g_k - H_k (x^* - x_k)\|_2 \leq \frac{\gamma}{2} \|x_k - x^*\|_2$$

which implies

$$\|x_{k+1} - x^*\|_2 \leq \frac{\gamma}{2} \|H_k^{-1}\|_2 \|x_k - x^*\|_2 \leq \frac{\gamma}{\lambda_{\text{min}}(H(x^*))} \|x_k - x^*\|_2 \tag{30}$$

Thus, we know that for $k \in \mathcal{K}$ sufficiently large, the unit step will be accepted and

$$\|x_{k+1} - x^*\|_2 \leq \frac{\gamma}{2} \|x_k - x^*\|_2$$

which proves parts (i) and (ii) since the entire argument may be repeated. Finally, part (iii) then follows from part (ii) and (30) with the choice

$$\kappa = \frac{\gamma}{\lambda_{\text{min}}(H(x^*))}$$

which completes the proof.
Figure: Contours for the objective function $f(x, y) = 10(y - x^2)^2 + (x - 1)^2$, and the iterates generated by the modified-Newton backtracking-Armijo Algorithm 15.

Algorithm 16: Modified-Newton Wolfe linesearch method

1: Input: initial guess $x_0$
2: Set $k = 0$
3: until $\|\nabla f(x_k)\| \leq 10^{-8} \max(1, \|\nabla f(x_0)\|)$
4: Compute positive-definite matrix $B_k$ from $H_k$ using Algorithm 2.
5: Compute a search direction $p_k$ as the solution to $B_k p_k = -\nabla f(x_k)$.
6: Compute a stepsize $\alpha_k$ that satisfies the Wolfe conditions (13a) and (13b).
7: Set $x_{k+1} \leftarrow x_k + \alpha_k p_k$
8: Set $k \leftarrow k + 1$
9: end until
10: return approximate solution $x_k$

- should also terminate if a maximum number of allowed iterations is reached
- should also terminate if a maximum time limit is reached

Notes
Algorithm 17  Quasi-Newton Wolfe linesearch method

1: Input: initial guess $x_0$
2: Set $k = 0$
3: until $\|\nabla f(x_k)\| \leq 10^{-8} \max (1, \|\nabla f(x_0)\|)$
4: Compute positive-definite matrix $B_k$ from $B_{k-1}$ using the BFGS rule, or using Algorithm 3 with a seed matrix $B_0$.
5: Compute a search direction $p_k$ as the solution to $B_k p_k = -g_k$.
6: Compute a stepsize $\alpha_k$ that satisfies the Wolfe conditions (13a) and (13b).
7: Set $x_{k+1} \leftarrow x_k + \alpha_k p_k$
8: Set $k \leftarrow k + 1$
9: end until
10: return approximate solution $x_k$

- should also terminate if a maximum number of allowed iterations is reached
- should also terminate if a maximum time limit is reached

Theorem 4.7 (Global convergence of modified-Newton/quasi-Newton Wolfe)

Suppose that
- $f \in C^1$ 
- $g$ is Lipschitz continuous on $\mathbb{R}^n$
- the condition number of $B_k$ is uniformly bounded by $\beta$

Then, the iterates generated by the modified/quasi-Newton Wolfe Algorithm 17 or 16 satisfy one of the following:

- finite termination, i.e., there exists a finite $k$ such that $g_k = 0$
- objective is unbounded below, i.e., $\lim_{k \to \infty} f_k = -\infty$
- convergence of gradients, i.e., $\lim_{k \to \infty} g_k = 0$

Proof:
Homework assignment.
Theorem 4.8 (Global convergence rate for modified-Newton/quasi-Newton Wolfe)

Suppose \( f \in C^1 \) and \( \nabla f \) is Lipschitz continuous on \( \mathbb{R}^n \). Further assume that \( f(x) \) is bounded from below on \( \mathbb{R}^n \). Let \( \{x_k\} \) be the iterates generated by the modified/quasi-Newton Wolfe linealsearch Algorithms 16 or 17. Then there exists a constant \( M \) such that for all \( T \geq 1 \)

\[
\min_{k=0,\ldots,T} \|\nabla f(x_k)\| \leq \frac{M}{\sqrt{T+1}}.
\]

Consequently, for any \( \epsilon > 0 \), within \( \lceil \left( \frac{M}{\epsilon} \right)^2 \rceil \) steps, we will see an iterate where the gradient has norm at most \( \epsilon \). In other words, we reach an "\( \epsilon \)-stationary" point in \( O\left( \left( \frac{1}{\epsilon} \right)^2 \right) \) steps.

Theorem 4.9 (Local superlinear convergence for quasi-Newton)

Suppose that \( f \in C^2 \) and \( H \) is Lipschitz continuous on \( \mathbb{R}^n \).

Suppose that iterates are generated from the quasi-Newton Wolfe Algorithm Algorithm 17 such that \( 0 < c_1 < c_2 \leq \frac{1}{2} \).

If the sequence of iterates \( \{x_k\} \) has a limit point \( x^* \) satisfying \( H(x^*) \) is positive definite and \( g(x^*) = 0 \), then it follows that

(i) the step length \( \alpha_k = 1 \) is valid for all sufficiently large \( k \).
(ii) the entire sequence \( \{x_k\} \) converges to \( x^* \) superlinearly.

See also Theorems 3.6 and 3.7 from Nocedal-Wright.
A Summary

**The problem**

\[
\begin{align*}
\text{minimize } & \ f(x) \\
\text{subject to } & \ x \in \mathbb{R}^n
\end{align*}
\]

where the objective function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \)

Solve instead quadratic approximation at current iterate \( x_k \), after choosing positive definite \( B_k \), to get a search direction that is a descent direction:

\[
p_k = \arg\min_{p \in \mathbb{R}^n} \quad m_k^G(p) \quad \text{def} = f_k + g_k^T p + \frac{1}{2} p^T B_k p
\]

- \( B_k \) = identity matrix: Steepest Descent Direction.
- \( B_k \) obtained from Hessian \( H_k \) by adjusting eigenvalues: Modified-Newton.
- \( B_k \) obtained by updating at every iteration using current and previous gradient information: Quasi-Newton.

Decide on step size strategy: Armijo backtracking line search, Wolfe backtracking line search, constant step size (HW assignment), \( O \left( \frac{\sqrt{k}}{k} \right) \) in iteration \( k \), many others.

**A Summary**

<table>
<thead>
<tr>
<th>Steepest Descent</th>
<th>Modified/Quasi Newton</th>
</tr>
</thead>
<tbody>
<tr>
<td>Requires only function and gradient evaluations</td>
<td>Could need Hessian evaluations as well (with spectral decompositions)</td>
</tr>
<tr>
<td>No overhead of solving linear system in each iteration</td>
<td>Solves a linear system ( Bp = g ) in each iteration</td>
</tr>
<tr>
<td>Global convergence to ( \epsilon )-stationary point in ( O(\epsilon^{-2}) ) steps</td>
<td>Global convergence to ( \epsilon )-stationary point in ( O(\epsilon^{-3}) ) steps</td>
</tr>
<tr>
<td>Linear local convergence: converges to ( \epsilon ) distance from local minimum in ( O(\log(\epsilon)) ) steps</td>
<td>Superlinear (quadratic) local convergence: converges to ( \epsilon ) distance from local minimum in ( o(\log(\epsilon)) ) ( (O(\log \log(\epsilon))) ) steps.</td>
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