Exercise 1.1: Compute $\nabla f(x)$ and $\nabla^2 f(x)$ for the following functions $f : \mathbb{R}^n \to \mathbb{R}$.

(a) $f(x) = \frac{1}{2} x^T H x$, where $H \in \mathbb{R}^{n \times n}$ is a fixed matrix. What if $H$ is symmetric?

(b) $f(x) = b^T A x - \frac{1}{2} x^T A^T A x$, where $A \in \mathbb{R}^{m \times n}$ is a fixed matrix and $b \in \mathbb{R}^m$ is a fixed vector.

(c) $f(x) = \|x\|_2 = (\sum_{i=1}^n x_i^2)^{1/2}$

(d) $f(x) = \|Ax - b\|_2$, where $A \in \mathbb{R}^{m \times n}$ is a fixed matrix and $b \in \mathbb{R}^m$ is a fixed vector.

Exercise 1.2: Let $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$. Let $x \in S$ and $s \in \mathbb{R}^n$ be such that $[x, x + s] \in S$.

(a) By defining $\phi(\alpha) = f(x + \alpha s)$ and using the Fundamental Theorem of Calculus:

$$\phi(1) = \phi(0) + \int_0^1 \phi'(\alpha) d\alpha,$$

show that

$$|f(x + s) - f(x) - g(x)^T s| \leq \frac{1}{2} \gamma_L \|s\|^2_2$$

whenever $f$ has a Lipschitz continuous gradient with Lipschitz constant $\gamma_L$ on $S$.

(b) Justify the formula

$$\phi(1) = \phi(0) + \phi'(0) + \int_0^1 \int_0^\alpha \phi''(t) \, dt \, d\alpha.$$

Hence, show that

$$|f(x + s) - f(x) - g(x)^T s - \frac{1}{2} s^T H(x) s| \leq \frac{1}{6} \gamma_Q \|s\|^3_2,$$

whenever $f$ has a Lipschitz continuous Hessian with Lipschitz constant $\gamma_Q$ on $S$.

Exercise 1.3: Write a MATLAB m-function that performs Newton’s Method for finding a zero of a function $F : \mathbb{R}^n \to \mathbb{R}^n$. The function call should have the form

$$[x,F,J,iter,status] = \text{newton}( \text{Fun},x0,maxit,printlevel,tol )$$

where $\text{Fun}$ is of type string that holds the name of a Matlab m-function, $x0$ is an initial guess at a zero, $\text{maxit}$ is the maximum number of iterations allowed, $\text{printlevel}$ determines the amount of printout required, and $\text{tol}$ is the final stopping tolerance. The Matlab m-function $\text{Fun}$ should have the form

$$[F,J] = \text{Fun}( x )$$
where \( F \) and \( J \) should contain the value and Jacobian of a desired function at the point \( x \). In the code, if the parameter `printlevel` has the value zero, then no printing should occur; otherwise, a single line of output is printed (in column format) per iteration. On output, the parameters \( x, F, \) and \( J \) should contain the final iterate, function value, and Jacobian matrix computed by the algorithm. The parameter `iter` should contain the total number of iterations performed. Finally, `status` should have the value 0 if the final stopping tolerance was obtained and the value 1 otherwise.

**Exercise 1.4** Let \( A \) be a given real symmetric matrix.

(a) Define an iteration of Newton’s Method for solving the \( n + 1 \) nonlinear equations

\[
(A - \lambda I)x = 0 \quad \text{and} \quad x^T x = 1
\]

in the \( n + 1 \) unknowns \((x, \lambda)\). Note that a zero \((x, \lambda)\) is an eigenpair of the matrix \( A \).

(b) Use the code you wrote for Exercise 2.1 to find an eigenpair \((x, \lambda)\) for the matrix

\[
A = \begin{pmatrix}
4 & 2 & 1 \\
2 & 3 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]

with starting point \( x_0 = \begin{pmatrix} 1/5 \\ -1/5 \\ 4/5 \end{pmatrix} \) and \( \lambda_0 = 1 \).