There are 7 questions on this test.

You have to return the test to me or Joe by 4:30 pm on Tuesday, October 27, 2015. No answers will be accepted after this deadline. You can choose to send your answers back electronically before that deadline or hand them back at the beginning of class on Tuesday, October 27.

Please hand in ONE submission - multiple submissions will not be tolerated.

You are not allowed to discuss any problem with another human being, except Dr. Basu.

You can use a computer only as a word processor; in particular, you cannot consult the internet in regards to this midterm. You CAN use any other resource like the textbook, your notes, books from the library.

You CAN cite any result we have mentioned in class or from the HWs without proof. If you cite a result (e.g., from a book) that was NOT mentioned in class, you should include a complete proof of this fact.

The level of rigor expected is the same as the HW solutions. Make sure you justify all your answers.

There will be partial credit; e.g., if you do not prove part (i) of some question, but use part (i)’s result to prove part (ii), you can get full credit for part (ii).

1. (20 pts) We say that a set \( X \subseteq \mathbb{R}^n \) is bounded if and only if there exists a number \( M \geq 0 \) such that \( X \subseteq \{ x \in \mathbb{R}^n : -M \leq x_i \leq M \ \forall i = 1, \ldots, n \} \).

Let \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) where \( A \) is an \( m \times n \) matrix and \( b \in \mathbb{R}^m \) and assume that \( P \) is nonempty – this assumption was added on Monday, Oct. 26 at 2:45pm. Define the set \( C := \{ r \in \mathbb{R}^n : Ar \leq 0 \} \). Show that \( P \) is bounded if and only if \( C = \{0\} \).

Hint: \( P \) is bounded if and only if the linear programs max\( \{ x_i : Ax \leq b \} \) and max\( \{ -x_i : Ax \leq b \} \) have finite values for all \( i = 1, \ldots, n \).

2. Let \( B \) denote the set of inessential vertices of \( G \). Let \( C = N(B) \cap (V \setminus B) \) where \( N(B) \) denotes the neighborhood of \( B \). Let \( D = V \setminus (B \cup C) \). Show that

i. (10 pts) \( C \) is a minimizer of the RHS of the Tutte-Berge formula.

ii. (5 pts) For every maximum matching \( M \), and every vertex \( v \) in \( C \), there is an edge \( vw \in M \) with \( w \in B \).

iii. (5 pts) Every maximum matching creates a perfect matching on the graph induced by \( D \).

3. (5 pts) Let \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) where \( A \) is an \( m \times n \) matrix and \( b \in \mathbb{R}^m \). Let \( c \in \mathbb{R}^n \) and suppose \( \delta = \max \{ c^T x : x \in P \} \). Define \( F = \{ x \in P : c^T x = \delta \} \), i.e., \( F \) is the set of ALL optimal solutions to the linear program. Show that if \( x \in F \) and \( x_1, x_2 \in P \) such that \( x = \frac{1}{2}(x_1 + x_2) \), then \( x_1, x_2 \in F \).
4. (20 pts) Find an efficient (polynomial time) algorithm for the following problem. We have a set of jobs $J_1, J_2, \ldots, J_k$, where each job has an associated nonnegative revenue $r_i, i = 1, \ldots, k$. We have a set of resources $T_1, \ldots, T_\ell$ where each resource has a nonnegative cost $c_j, j = 1, \ldots, \ell$. Each job $J_i$ requires a subset $S_i \subseteq \{T_1, \ldots, T_\ell\}$ of resources. However, if a resource $T_j$ is purchased, then this resource is available for any job that requires it - we only pay a one-time cost of $c_j$. Choose an optimum set of jobs to finish to maximize the total associated revenue minus the total cost of the resources required.

5. (20 pts) Let $A$ be an $m \times n$ matrix such that every entry is either 0 or 1. Moreover, in any column the 1’s are consecutive, i.e., every column is of the following form:

$$[0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0]^T.$$  

Show that $A$ is totally unimodular.

Hint: Use row operations to reduce to a matrix discussed in class.

6. (5 pts) Let $A$ be an $p \times n$ matrix and $C$ be a $q \times n$ matrix. Let $b \in \mathbb{R}^p$ and $d \in \mathbb{R}^q$. Let $c \in \mathbb{R}^n$. Show that

$$\max_{x \in \mathbb{R}^n} \{c^T x : Ax \leq b, Cx = d\} = \min_{y \in \mathbb{R}^p, w \in \mathbb{R}^q} \{y^T b + w^T d : y^T A + w^T C = c^T, y \geq 0\}$$

assuming that both the right hand side and left hand side are finite.

7. (10 pts) Let $G = (V, E)$ be an undirected graph; $n = |V|, m = |E|$. For each edge $e \in E$, let $u_e$ be a nonnegative weight associated with it. For any subset $S \subseteq V$, let $\delta(S) = \{e = uv \in E : u \in S, v \notin S\}$ and define $u(\delta(S)) = \sum_{e \in \delta(S)} u_e$. Devise an algorithm whose runtime is a polynomial in $n, m$ to find

$$S^* = \arg \min_{\emptyset \subsetneq S \subseteq V} u(\delta(S)),$$

i.e., find a nonempty, proper subset $S \subseteq V$ that minimizes $u(\delta(S))$. 