There are 7 questions on this test, some with subparts.

You have to return the test electronically to me or Tu Nguyen by 12:00 noon on Thursday, May 7, 2020. No answers will be accepted after this deadline.

Please hand in ONE submission - multiple submissions will not be tolerated.

You are not allowed to discuss any problem with another human being, except the instructor for the course.

You can use a computer only as a word processor; in particular, you cannot consult the internet in regards to this final. You CAN use any other resource like the textbook, your notes, books from the library.

You CAN cite any result we have mentioned in class or from the HWs without proof. If you cite a result (e.g., from a book) that was NOT mentioned in class, you should include a complete proof of this fact.

The level of rigor expected is the same as the HW solutions. Make sure you justify all your answers.

There will be partial credit; e.g., if you do not prove part (i) of some question, but use part (i)’s result to prove part (ii), you can get full credit for part (ii).

I attest that I have completed this exam without unauthorized assistance from any person, materials, or device.

Name ____________________________________________
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1. (20 pts) Give a polynomial time algorithm for solving the following problem in matrices. Let $U = (u_{ij})$ be a fixed $n \times n$ matrix with nonnegative integer entries. Let $r_1, r_2, \ldots, r_n, c_1, c_2, \ldots, c_n$ be $2n$ positive integers. Determine if there exists an $n \times n$ matrix $A = (a_{ij})$ with nonnegative, integer entries such that all of the following hold:

(i) For each $i = 1, \ldots, n$, the sum of the entries of row $i$ is at most $r_i$.
(ii) For each $j = 1, \ldots, n$, the sum of the entries in column $j$ is exactly $c_j$.
(iii) $A \leq U$, i.e., $a_{ij} \leq u_{ij}$ for each $i, j \in \{1, \ldots, n\}$.

Justify the correctness and the polynomial running time of your algorithm.

2. (i) (10 pts) Show that the lattice generated by $(1, 0)$ and $(1/2, 1)$ does not have an orthogonal basis.

(ii) (10 pts) Let $\Lambda \subseteq \mathbb{R}^n$ be a lattice such that the span of $\Lambda$ is all of $\mathbb{R}^n$. Consider the following set

$\Lambda^* := \{w \in \mathbb{R}^n : \langle w, z \rangle \in \mathbb{Z} \ \forall z \in \Lambda\}$.

Prove that $\Lambda^*$ is a lattice as well. [In class, we had called $\Lambda^*$ the dual lattice, but never explicitly verified it is a lattice. I want you to do this here.]

3. Stable Set Polytope. Consider the graph in the figure below on the vertex set $V = \{1, 2, \ldots, 7\}$ and the integer program for the independent set/stable set problem in that graph:

$$\begin{align*}
\max & \quad \sum_{i=1}^{7} x_i \\
\text{subject to} & \quad x_i + x_j \leq 1 \ \forall i, j \in V \text{ such that } ij \in E \\
& \quad x_i \geq 0 \ \forall i \in V \\
& \quad x_i \in \mathbb{Z} \ \forall i \in V
\end{align*} \tag{1}$$

(i) (7 pts) Give a combinatorial argument that the inequality $\sum_{i=1}^{7} x_i \leq 2$ is a valid inequality for the feasible points of the integer program.

(ii) (8 pts) Find a cutting plane proof of the inequality $\sum_{i=1}^{7} x_i \leq 2$ using sequences of C-G cuts (starting from the system (1)).

4. (10 pts) Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron and let $v \in P$ be a vertex of $P$ such that $v \notin \mathbb{Z}^n$. Show that there always exists a Chvátal-Gomory cutting plane $H$ for $P$ such that $v \notin H$. 

3
5. (10 pts) Let $n$ be any natural number. Consider the undirected graph $G_n = (V_n, E_n)$, where $V_n$ is the set of all 0–1 vectors in $\mathbb{R}^n$. Two vectors $u, v \in V$ are adjacent in $G_n$ if and only if $u$ and $v$ differ in exactly one coordinate. [Example: $n = 2$, we obtain the graph with four vertices $(0, 0), (0, 1), (1, 1), (1, 0)$, and four edges: {{0, 0}, (0, 1)}, {{0, 1}, (1, 1)}, {{1, 1}, (1, 0)}, {{1, 0}, (0, 0)}.] Compute the maximum matching and minimum vertex cover of $G_n$ in terms of $n$. Justify your answer.

6. Minimum Spanning Trees. Given a connected, undirected graph $G = (V, E)$, a spanning tree is a connected subgraph $T = (V, E' \subseteq E)$ (i.e., every pair of vertices from $V$ has a path connecting them in $T$) such that $T$ has no cycles. Given weights $w_e$ on each edge $e \in E$, the minimum spanning tree problem is to find a spanning tree with minimum total weight. Consider the following integer program defined over variables $x_e, e \in E$:

$$
\begin{align*}
\text{min} & \quad \sum_{e \in E} w_e x_e \\
\text{subject to} & \quad \sum_{e \in E} x_e = |V| - 1 \\
& \quad \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \subseteq V \text{ such that } 1 \leq |S| \leq |V| - 1 \\
& \quad 0 \leq x_e \leq 1 \quad \forall e \in E \\
& \quad x_e \in \mathbb{Z} \quad \forall e \in E
\end{align*}
$$

(i) (5 pts) Show that the above is an integer programming formulation for the 0-1 embedding of the spanning tree problem. More precisely, show that for every spanning tree $T$, its 0-1 embedding $x^T$ is a feasible solution to the above integer program, and vice versa, all integer solutions are 0-1 embeddings of some spanning tree. [You may use, without proof, the fact that every tree on $n$ vertices has $n - 1$ edges.]

(ii) (10 pts) Give an example of a graph $G = (V, E)$ for which the linear programming relaxation (dropping the integrality constraints $x_e \in \mathbb{Z}$) of the above integer program has a non-integral vertex. Justify your answer.

(iii) (5 pts) Give a polynomial time separation oracle for the linear programming relaxation for the above integer program (which has exponentially many constraints).

7. (5 pts) In a graph $G = (V, E)$, a dominating set is a subset of vertices $D \subseteq V$ such that every vertex NOT in $D$ is adjacent to a vertex in $D$. Formulate a integer linear program to find the smallest size dominating set in a graph.