AMS 553.766: Combinatorial Optimization
Homework Problems - Week IX

1. Solve the following integer program using Gomory’s fractional cutting planes.

\[
\begin{align*}
\text{max} & \quad 2x_1 + x_2 \\
\text{subject to} & \quad -x_1 + x_2 \leq 0 \\
& \quad 6x_1 + 2x_2 \leq 21 \\
& \quad x_1, x_2 \geq 0 \\
& \quad x_1, x_2 \in \mathbb{Z}
\end{align*}
\]

[Exercise 1.7 in the “Integer Programming” book by Conforti, Cornuéjols and Zambelli]

2. Cut generating functions

Recall the family of functions \( \phi_f : \mathbb{R} \to \mathbb{R} \) defined in class parameterized by \( f \in (0,1) \):

\[
\phi_f(r) = \begin{cases} 
\{r\} & \text{if } \{r\} \leq f \\
1 - \{r\} & \text{if } \{r\} > f
\end{cases}
\]

(i) Show that each \( \phi_f \) is subadditive.

(ii) Show that \( \phi_f(-r) = \phi_{1-f}(r) \). (This means the family is closed under reflection around the \( y \) axis)

(iii) Consider the function \( \psi : \mathbb{R} \to \mathbb{R} \) defined as

\[
\psi(r) = \begin{cases} 
\sin(\pi\{r\}) & \text{if } 0 \leq \{r\} < \frac{1}{2} \\
2 - \sin(\pi(1 - \{r\})) & \frac{1}{2} \leq \{r\} < 1
\end{cases}
\]

Show that \( \psi \) is periodic with period 1 and subadditive.

3. Cutting planes from disjunctions

Recall that if \( P \subseteq \mathbb{R}^n \) is a polyhedron, \( D = Q_1 \cup \ldots \cup Q_t \) is a valid disjunction (i.e., \( \mathbb{Z}^n \subseteq D \)), then a halfspace \( H \) is said to be a cutting plane for \( P \) derived from \( D \) if \( P \cap D \subseteq H \).

Let \( P \subseteq \mathbb{R}^n \) be a polyhedron and let \( v \in P \) be a vertex of \( P \) such that \( v \notin \mathbb{Z}^n \). Show that there exists a valid two-term disjunction of halfspaces, i.e., \( D = Q_1 \cup Q_2 \) where \( Q_1, Q_2 \) are both halfspaces, and a cutting plane \( H \) for \( P \) derived from \( D \) such that \( v \notin H \).

Does the statement remain necessarily true if \( v \in P \setminus \mathbb{Z}^n \) but \( v \) is not a vertex of \( P \)?