1. Let \((X, \mathcal{I})\) be an independence system. Show that \((X, \mathcal{I})\) is a matroid if and only if for every \(Z \subseteq X\) all bases of \(Z\) are of the same cardinality. [We proved the “only if” direction in class]

Solution: \((\Rightarrow)\) Consider an arbitrary \(Z \subseteq X\). Let \(B_1, B_2\) be bases of \(Z\). If \(|B_1| < |B_2|\), then by the matroid condition, there exists \(x \in B_2 \setminus B_1\) such that \(B_1 \cup \{x\} \in \mathcal{I}\). Since \(x \in B_2 \subseteq Z\), this would mean that \(B_1 \cup \{x\} \subseteq Z\) contradicting the maximality of \(B_1\).

\((\Leftarrow)\) Let \(F, F' \in \mathcal{I}\) such that \(|F| < |F'|\). Let \(Z = F \cup F'\). Now all bases of \(Z\) have the same size, and \(F' \subseteq Z\). Therefore, all bases of \(Z\) must have size at least \(|F'|\). Since \(F \in \mathcal{I}\), there must be a basis of \(Z\) containing \(F\) and it must have size at least \(|F'| > |F|\). This shows there must exist \(x \in Z \setminus F\) such that \(F \cup \{x\} \in \mathcal{I}\). Since \(Z = F \cup F'\), this must mean that \(x \in F' \setminus F\).

2. Let \((X, \mathcal{I})\) be a matroid. Let \(Z \subseteq X\) be any subset. Let \(X' = X \setminus Z\) and \(\mathcal{I}' = \{Y \setminus Z : Y \in \mathcal{I}\}\). Show that \((X', \mathcal{I}')\) is a matroid. Use this to show that if one can solve the maximum weight basis problem for matroids, then one can solve the maximum weight independence set problem for matroids.

Solution: Let \(F, F' \in \mathcal{I}'\) with \(|F| < |F'|\). By definition of \(\mathcal{I}'\), there must be sets \(F_1, F'_1\) such that \(F = F_1 \setminus Z\) and \(F' = F'_1 \setminus Z\). This means \(F \subseteq F_1\) and \(F' \subseteq F'_1\). Since \(\mathcal{I}\) is an independence system, this means that \(F, F' \in \mathcal{I}\). By applying the matroid condition, we obtain that there exists \(x \in F' \setminus F\) such that \(F \cup \{x\} \in \mathcal{I}\). Since \(x \in F'\) and \(F' \cap Z = \emptyset, x \notin Z\). Therefore, \((F \cup \{x\}) \setminus Z = F \cup \{x\}\). By definition, \(F \cup \{x\} \in \mathcal{I}'\) and we are done.

3. Let \(X\) be a finite set and let \(k\) be a natural number. Let \(\mathcal{I} := \{Y \subseteq X : |Y| \leq k\}\). Show that \((X, \mathcal{I})\) is a matroid. Such matroids are called \(k\)-uniform matroids.

Solution: Let \(F, F' \in \mathcal{I}\) with \(|F| < |F'| \leq k\). Then consider any element \(x \in F' \setminus F\), \(|F \cup \{x\}| = |F| + 1 \leq |F'| \leq k\). Therefore, by definition \(F \cup \{x\} \in \mathcal{I}\).

4. Let \(M = (X, \mathcal{I})\) be a matroid and let \(k\) be a natural number. Define \(\mathcal{I} := \{Y \in \mathcal{I} : |Y| \leq k\}\). Show that \((X, \mathcal{I})\) is again a matroid (called the \(k\)-truncation of \(M\)).

Solution: Let \(F, F' \in \mathcal{I}\) with \(|F| < |F'| \leq k\). Since \(M\) is a matroid, there exists \(x \in F' \setminus F\) such that \(F \cup \{x\} \in \mathcal{I}\). \(|F \cup \{x\}| = |F| + 1 \leq |F'| \leq k\). Therefore, by definition \(F \cup \{x\} \in \mathcal{I}'\).

5. Let \((X, \mathcal{I})\) be a matroid. Two elements \(x, y \in X\) are called parallel if \(\{x, y\}\) is a circuit. Show that if \(x\) and \(y\) are parallel and \(Y\) is an independent set with \(x \in Y\), then \((Y \setminus \{x\}) \cup \{y\}\) is also independent.

Solution: Let \(Z = Y \cup \{y\}\). Since \(\{x, y\}\) is a circuit, \(Z\) is a dependent set. Since \(Y \in \mathcal{I}\) and \(Y \subseteq Z\), this makes \(Y\) a basis of \(Z\). Consider a basis \(B\) of \(Z\) such that \(y \in B\). Thus, \(B \setminus \{y\} \subseteq Y\). Moreover, \(|B| = |Y|\) because all bases of \(Z\) have the same cardinality. This implies that \(|B \cap Y| = |Y| - 1\). Since \(x \notin B\) because \(\{x, y\}\) is a circuit, we have \(B \cap Y = Y \setminus \{x\}\) and so \(B = (B \cap Y) \cup \{y\} = Y \setminus \{x\} \cup \{y\}\).