AMS 550.666: Combinatorial Optimization
Homework Problems - Week IX

Consider a positive semidefinite \((n+1) \times (n+1)\) matrix \(X\), whose rows and columns are indexed by \(\{0, 1, \ldots, n\}\) as in class. We define \(\text{Proj}_x(X) = (X_{01}, X_{02}, \ldots, X_{0n}) \in \mathbb{R}^n\). For any vector \(v \in \mathbb{R}^d\), \(\text{Diag}(v)\) is a \(d \times d\) diagonal matrix with the diagonal entries equal to the corresponding entries of \(v\).

1. **Semidefinite Programs**

   (i) Let \(S_n \subseteq \mathbb{R}^{n^2}\) be the set of all \(n \times n\) positive semidefinite matrices. Show that \(S_n\) is a convex cone in \(\mathbb{R}^{n^2}\), i.e., for any PSD matrix \(A\), \(\lambda A\) is also PSD for all \(\lambda \geq 0\), and for any PSD matrices \(A, B\), \(A + B\) is also PSD.

   (ii) Show that any linear program of the form \(\max \{c^T x : Ax = b, x \geq 0\}\) can be solved by writing an equivalent semidefinite program.

2. **Rank constraints are not convex.** Show that the set
\[
\{X \in \mathbb{R}^{n^2} : X \text{ is PSD, } \text{rank}(X) \leq k\}
\]
is not a convex set for any \(1 \leq k < n\).

3. **Relaxations.**

   (i) **Stable Set.** Given a graph \(G = (V, E)\), with \(V = \{1, \ldots, n\}\), show that
\[
\text{STAB}(G) := \text{conv}(\left\{ x \in \mathbb{R}^n : x_i + x_j \leq 1 \quad \forall ij \in E \right\}) \subseteq \left\{ \text{Proj}_x(X) : \begin{array}{l}
X_{00} = 1, \\
X_{ii} = X_{i0} \quad \forall i = 1, \ldots, n, \\
X_{ij} = 0 \quad \forall ij \in E \\
X \text{ is PSD}
\end{array} \right\}
\]

   (ii) **Max-Cut.** Show that
\[
\max\left\{ \sum_{i,j=1}^{n} w_{ij} \frac{1 - x_i x_j}{2} : x_i \in \{1, -1\} \quad \forall i = 1, \ldots, n \right\} \leq \max\left\{ \sum_{i,j=1}^{n} w_{ij} \frac{1 - X_{ij}}{2} : X_{ii} = 1, \ X \text{ is PSD}\right\}.
\]

4. Consider the feasible set of the following integer program: \(\{x : Ax \leq b, x \in \{0,1\}\}\). We form the “naive” semidefinite program for it by
\[
\text{SDP} = \left\{ \text{Proj}_x(X) : \begin{array}{l}
X \text{ is PSD,} \\
X_{00} = 1, \\
X_{ii} = X_{i0} \quad \forall i = 1, \ldots, n, \\
A \cdot (\text{Proj}_x(X)) \leq b
\end{array} \right\}
\]
Show that \(\text{SDP} = \{x \in \mathbb{R}^n : Ax \leq b, 0 \leq x \leq 1\}\).

This shows that the naive SDP formulation of a \(0-1\) integer program is no better than the LP relaxation.