AMS 550.666: Combinatorial Optimization
Homework Problems - Week VIII

1. A company sets an auction for $N$ objects. Bidders place their bids for some subsets of the $N$ objects that they like. The auction house has received $n$ bids, namely bids $b_j$ for subset $S_j$, for $j = 1, \ldots, n$. The auction house is faced with the problem of choosing the winning bids so that profit is maximized and each of the $N$ objects is given to at most one bidder. Formulate the optimization problem faced by the auction house as an integer programming problem.

[Adapted from Problem 2.16 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

2. Jobs $\{1, \ldots, n\}$ must be processed on a single machine. Each job is available for processing after a certain time, called release time. For each job we are given its release time $r_i$, its processing time $p_i$ and its weight $w_i$. Formulate as an integer linear program the problem of sequencing the jobs without overlap or interruption so that the sum of the weighted completion times is minimized.

[Problem 2.17 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

3. A firm is considering project $A, B, \ldots, H$. Using binary variables $x_a, \ldots, x_h$ and linear constraints, model the following conditions on the projects to be undertaken.

(a) At most one of $A, B, \ldots, H$.
(b) Exactly two of $A, B, \ldots, H$.
(c) $A$ or $B$.
(d) $A$ and $B$.
(e) If $A$ then $B$.
(f) If $A$ then not $B$.
(g) If not $A$ then $B$.
(h) If $A$ then $B$, and if $B$ then $A$.
(i) If $A$ then $B$ and $C$.
(j) If $A$ then $B$ or $C$.
(k) If $B$ or $C$ then $A$.
(l) If $B$ and $C$ then $A$.
(m) If two or more of $B, C, D, E$ then $A$.
(n) If $m$ or more than $n$ projects $B, \ldots, H$ then $A$.

[Problem 2.19 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

4. For the following subsets of edges of an undirected graph $G = (V, E)$, we view the following sets as $(0, 1)$ vectors in $\mathbb{R}^{|E|}$ in the standard way. Find an integer programming formulation and prove its correctness:

(a) The family of Hamiltonian paths of $G$ with endnodes $u, v$. (A Hamiltonian path is a path that goes exactly once through each node/vertex of the graph.)
(b) The family of all Hamiltonian paths of $G$.
(c) The family of edge sets that induce a triangle of $G$. 
(d) Assuming that $G$ has $3n$ nodes, the family of $n$ node-disjoint triangles.

(e) The family of odd cycles of $G$.

Research question: For each problem above, is it possible to find a formulation using polynomially many inequalities (in the size of the graph $G$), or show that no such formulation exists?

[Problem 2.21 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

5. (Playing with $(0,1)$-vectors). Find integer programming formulations for the following integer sets.

(a) The set of all $(0,1)$-vectors in $\mathbb{R}^4$ except \[
\begin{pmatrix}
0 \\
1 \\
1 \\
0
\end{pmatrix}.
\]

(b) The set of all $(0,1)$-vectors in $\mathbb{R}^6$ except \[
\left\{ \begin{pmatrix}
0 \\
1 \\
1 \\
0 \\
1 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0 \\
1 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix} \right\}.
\]

(c) The set of all $(0,1)$-vectors in $\mathbb{R}^6$ except all the vectors having exactly two 1s in the first 3 components and one 1 in the last 3 components.

(d) The set of all $(0,1)$-vectors in $\mathbb{R}^n$ with an even number of 1s. You don’t have to find a system with $\text{poly}(n)$ inequalities.

(e) The set of all $(0,1)$-vectors in $\mathbb{R}^n$ with an odd number of 1s. You don’t have to find a system with $\text{poly}(n)$ inequalities.

Research question: For problems (d) and (e), is it possible to find a formulation using $\text{poly}(n)$ many inequalities, or show that no such formulation exists?

[Adapted from Problem 2.27 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]