1. A company sets an auction for $N$ objects. Bidders place their bids for some subsets of the $N$ objects that they like. The auction house has received $n$ bids, namely bids $b_j$ for subset $S_j$, for $j = 1, \ldots, n$. The auction house is faced with the problem of choosing the winning bids so that profit is maximized and each of the $N$ objects is given to at most one bidder. Formulate the optimization problem faced by the auction house as an integer programming problem.

[Introduced from Problem 2.16 from the “Integer Programming” textbook by Conforti, Cornuèjols and Zambelli.]

Introduce a binary variable $x_j$ for each bidder $j = 1, \ldots, n$ indicating whether the bidder was assigned his subset or not. The problem can then be solved by the following integer program:

\[
\begin{align*}
\text{max} & \quad \sum_{j=1}^{n} b_j x_j \\
\text{subject to} & \quad \sum_{j \in S_i} x_j \leq 1 \quad i = 1, \ldots, N \\
& \quad x_i \leq 1 \quad i = 1, \ldots, N \\
& \quad x_i \geq 0 \quad i = 1, \ldots, N \\
& \quad x_i \in \mathbb{Z} \quad i = 1, \ldots, N
\end{align*}
\]

2. Jobs $\{1, \ldots, n\}$ must be processed on a single machine. Each job is available for processing after a certain time, called release time. For each job we are given its release time $r_i$, its processing time $p_i$ and its weight $w_i$. Formulate as an integer linear program the problem of sequencing the jobs without overlap or interruption so that the sum of the weighted completion times is minimized.

[Problem 2.17 from the “Integer Programming” textbook by Conforti, Cornuèjols and Zambelli.]

Solution: For every pair of $i < j$ of jobs, introduce a binary variable $x_{ij}$ which is 1 if job $i$ is processed before job $j$, and 0 otherwise. We also introduce a continuous variable $s_j$, $j = 1, \ldots, n$, which indicates the start time of job $j$.

The problem can then be solved by the following integer program:

\[
\begin{align*}
\text{max} & \quad \sum_{i=1}^{n} w_i (s_i + p_i) \\
\text{subject to} & \quad s_i \geq r_i \quad \forall \ i = 1, \ldots, n \\
& \quad s_i + p_i \leq s_j + p_j + M(1 - x_{ij}) \quad \forall \ i < j \\
& \quad s_j + p_j \leq s_i + p_i + M x_{ij} \quad \forall \ i < j \\
& \quad x_{ij} \leq 1 \quad \forall \ i < j \\
& \quad x_{ij} \geq 0 \quad \forall \ i < j \\
& \quad x_{ij} \in \mathbb{Z} \quad \forall \ i < j
\end{align*}
\]

where $M$ is a sufficiently large integer, such as $M = \sum_i (r_i + p_i)$.

3. A firm is considering project $A, B, \ldots, H$. Using binary variables $x_a, \ldots, x_h$ and linear constraints, model the following conditions on the projects to be undertaken.

(a) At most one of $A, B, \ldots, H$.
(b) Exactly two of $A, B, \ldots, H$.
(c) $A$ or $B$.
(d) $A$ and $B$.
(e) If $A$ then $B$. 

(f) If $A$ then not $B$.
(g) If not $A$ then $B$.
(h) If $A$ then $B$, and if $B$ then $A$.
(i) If $A$ then $B$ and $C$.
(j) If $A$ then $B$ or $C$.
(k) If $B$ or $C$ then $A$.
(l) If $B$ and $C$ then $A$.
(m) If two or more of $B, C, D, E$ then $A$.
(n) If $m$ or more than $n$ projects $B, . . . , H$ then $A$.

[Problem 2.19 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

Solution:

(a) $x_a + x_b + \ldots + x_h \leq 1$.
(b) $x_a + x_b + \ldots + x_h = 2$.
(c) $x_a + x_b \geq 1$.
(d) $x_a + x_b \geq 2$.
(e) $x_a \leq x_b$.
(f) $x_a + x_b \leq 1$.
(g) $x_a + x_b \geq 1$.
(h) $x_a = x_b$.
(i) $x_b + x_c \geq 2x_a$.
(j) $x_b + x_c \geq x_a$.
(k) $x_b \leq x_a, x_c \leq x_a$.
(l) $1 + x_a \geq x_b + x_c$.
(m) $1 + x_a \geq x_i + x_j$, for all $i, j \in \{b, c, d, e\}$.
(n) $n - 1 + x_a \geq \sum_{i \in S} x_i$ \quad $\forall$ $S \subseteq \{B, C, \ldots, H\}$ such that $|S| = n$

4. For the following subsets of edges of an undirected graph $G = (V, E)$, we view the following sets as $(0, 1)$ vectors in $\mathbb{R}^{|E|}$ in the standard way. Find an integer programming formulation and prove its correctness:

(a) The family of Hamiltonian paths of $G$ with endnodes $u, v$. (A Hamiltonian path is a path that goes exactly once through each node/vertex of the graph.)

Solution: $\sum_{e \in \delta(x)} x_e = 2$ for all $x \neq u, v$,
$\sum_{e \in \delta(x)} x_e = 1$ for all $x = u, v$,
$\sum_{e \in \delta(S)} x_e \geq 1$ for all $S \subseteq V$.

(b) The family of all Hamiltonian paths of $G$.

Solution: $\sum_{e \in \delta(x)} x_e \leq 2$ for all $x \neq u, v$,
$\sum_{e \in E} x_e = n - 1$,
$\sum_{e \in \delta(S)} x_e \geq 1$ for all $S \subseteq V$.

The last two set of constraints impose the condition that we have a spanning tree: the second constraint imposes that we have $n - 1$ edges, and the third constraint imposes that we have a connected subgraph.
(c) The family of edge sets that induce a triangle of $G$.
Solution:
\[ \sum_{e \in E} x_e = 3, \]
For every triangle $T$ in $G$ with edges $x_i, x_j, x_k$, put the constraint $1 + x_i \geq x_j + x_k$ with all permutations of $i, j, k$.

(d) Assuming that $G$ has $3n$ nodes, the family of $n$ node-disjoint triangles.
Solution:
\[ \sum_{e \in E} x_e = 3n, \]
For every triangle $T$ in $G$ with edges $x_i, x_j, x_k$, put the constraint $1 + x_i \geq x_j + x_k$ with all permutations of $i, j, k$.

(e) The family of odd cycles of $G$.
Solution: Observe that for any $0-1$ vector $y \in \mathbb{R}^n$, the constraint \[ \sum_{i=1}^{n} (y_i - \frac{1}{2})(x_i - \frac{1}{2}) \leq n/4, \] as a constraint on the variables $x_1, \ldots, x_n$, is satisfied by all $0-1$ vectors except $x = y$. Thus, for every $0-1$ vector $y \in \mathbb{R}^{|E|}$ that is not an odd cycle, we impose the constraint that \[ \sum_{e}(y_e - \frac{1}{2})(x_e - \frac{1}{2}) \leq |E|/4. \]

Research question: For each problem above, is it possible to find a formulation using polynomially many inequalities (in the size of the graph $G$), or show that no such formulation exists?
[Problem 2.21 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

5. (Playing with $(0,1)$-vectors). Find integer programming formulations for the following integer sets.

(a) The set of all $(0,1)$-vectors in $\mathbb{R}^4$ except \[
\begin{pmatrix}
0 \\
1 \\
1 \\
0 
\end{pmatrix}.
\]

(b) The set of all $(0,1)$-vectors in $\mathbb{R}^6$ except \[
\left\{
\begin{pmatrix}
0 \\
1 \\
1 \\
0 \\
1 \\
0 
\end{pmatrix}, 
\begin{pmatrix}
0 \\
1 \\
1 \\
0 \\
1 \\
1 
\end{pmatrix}, 
\begin{pmatrix}
1 \\
0 \\
1 \\
0 \\
1 \\
1 
\end{pmatrix}
\right\}.
\]

(c) The set of all $(0,1)$-vectors in $\mathbb{R}^6$ except all the vectors having exactly two 1s in the first 3 components and one 1 in the last 3 components.

(d) The set of all $(0,1)$-vectors in $\mathbb{R}^n$ with an even number of 1s. You don’t have to find a system with $\text{poly}(n)$ inequalities.

(e) The set of all $(0,1)$-vectors in $\mathbb{R}^n$ with an odd number of 1s. You don’t have to find a system with $\text{poly}(n)$ inequalities.

Research question: For problems (d) and (e), is it possible to find a formulation using $\text{poly}(n)$ many inequalities, or show that no such formulation exists?
[Adapted from Problem 2.27 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

Solution: All of the above problems can be solved using the principle observed in Problem 4(e) above: For any $0-1$ vector $y \in \mathbb{R}^n$, the constraint \[ \sum_{i=1}^{n} (y_i - \frac{1}{2})(x_i - \frac{1}{2}) \leq n/4, \] as a constraint on the variables $x_1, \ldots, x_n$, is satisfied by all $0-1$ vectors except $x = y$. 
