1. Let $G = (V, E)$ be a graph. Consider variables $x_e$ associated with each edge $e \in E$. Let $P(G)$ be the polytope given by:

$$
\sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \\
\sum_{e \in E} x_e \geq 0 \quad \forall e \in E
$$

We showed in class that if $G$ is bipartite, then all vertices of $P(G)$ are integral by the TUM property of the constraint matrix. Show the converse that if $G$ is not bipartite, then there exists a non integral vertex.

2. (i) Let $A \in \mathbb{R}^{m \times n}$ be a matrix with rational entries and $b \in \mathbb{R}^m$ be a vector with rational coordinates. Show that every vertex of the polyhedron \{ $x \in \mathbb{R}^n$ : $Ax \leq b$ \} has rational coordinates. [Hint: Use the characterization of vertices in terms of the rank of the tight constraints.]

(ii) Let $p^1, \ldots, p^k, v \in \mathbb{R}^n$ be vectors with rational coordinates such that $v$ is in the convex hull of \{ $p^1, \ldots, p^k$ \}. Show that there exists rational coefficients $\lambda_1, \ldots, \lambda_k \geq 0$ with $\sum_i \lambda_i = 1$ and $v = \lambda_1 p^1 + \ldots + \lambda_k p^k$. [Hint: Show that the set of all such coefficients forms a polyhedron in $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ space and use part (i).]

3. In class, we showed that given a graph $G$, the convex hull of all 0/1 perfect matching vectors is given by

$$
Q(G) := \left\{ x \in \mathbb{R}^E : \begin{array}{l}
\sum_{e \in \delta(v)} x_e = 1 \quad \forall v \in V \\
\sum_{e \in E[S]} x_e \leq \frac{|S| - 1}{2} \quad \forall S \subseteq V \text{ such that } |S| \text{ is odd} \\
x_e \geq 0 \quad \forall e \in E
\end{array} \right\}
$$

Recall the construction from class that reduces the general matching problem to a perfect matching problem: Given any graph $G$, one creates a new graph $\tilde{G}$ by making two copies of $G$ and adding edges between corresponding vertices in the two copies. Recall also that we defined $P_{\text{matching}}(G)$ be the polyhedron obtained by replacing the equality constraints above by $\sum_{e \in \delta(v)} x_e \leq 1$. Finally, we related $P_{\text{matching}}(G)$ and $Q(\tilde{G})$ by the following construction: given $\bar{x} \in P_{\text{matching}}(G)$, we create $\tilde{x}$ by assigning the $x$ values on edges in the two copies of $G$ and for every vertex $v$ of $G$, assign $1 - \sum_{e \in \delta(v)} \bar{x}_e$ as the value of $\tilde{x}$ to the edge in $\tilde{G}$ joining the two copies of the vertex $v$.

Show that with this construction $\tilde{x} \in Q(\tilde{G})$.

4. Given an undirected graph, a clique is a subgraph that is complete, i.e. all pairs of vertices in the subgraph have an edge between them. In other words, a clique is given by a subset of the vertices that are all connected to each other by edges.

Write an integer program to find the largest clique (in terms of number of vertices) in a graph $G = (V, E)$. Let $G' = (V, E')$ be the graph on the same set of vertices $V$ such that $ij \in E'$ if and only if $ij \not\in E$. Interpret the dual of the LP relaxation of the largest clique IP for $G$ in terms of $G'$.

5. A company sets an auction for $N$ objects. Bidders place their bids for some subsets of the $N$ objects that they like. The auction house has received $n$ bids, namely bids $b_j$ for subset $S_j$, for $j = 1, \ldots, n$. The auction house is faced with the problem of choosing the winning bids so
that profit is maximized and each of the \( N \) objects is given to at most one bidder. Formulate the optimization problem faced by the auction house as an integer programming problem.

[Adapted from Problem 2.16 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

6. Jobs \( \{1, \ldots, n\} \) must be processed on a single machine. Each job is available for processing after a certain time, called release time. For each job we are given its release time \( r_i \), its processing time \( p_i \) and its weight \( w_i \). Formulate as an integer linear program the problem of sequencing the jobs without overlap or interruption so that the sum of the weighted completion times is minimized.

[Problem 2.17 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

7. A firm is considering project \( A, B, \ldots, H \). Using binary variables \( x_a, \ldots, x_h \) and linear constraints, model the following conditions on the projects to be undertaken.

   (a) At most one of \( A, B, \ldots, H \).
   (b) Exactly two of \( A, B, \ldots, H \).
   (c) \( A \) or \( B \).
   (d) \( A \) and \( B \).
   (e) If \( A \) then \( B \).
   (f) If \( A \) then not \( B \).
   (g) If not \( A \) then \( B \).
   (h) If \( A \) then \( B \), and if \( B \) then \( A \).
   (i) If \( A \) then \( B \) and \( C \).
   (j) If \( A \) then \( B \) or \( C \).
   (k) If \( B \) or \( C \) then \( A \).
   (l) If \( B \) and \( C \) then \( A \).
   (m) If two or more of \( B, C, D, E \) then \( A \).
   (n) If more than \( n \) projects \( B, \ldots, H \) then \( A \).

[Problem 2.19 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

8. For the following subsets of edges of an undirected graph \( G = (V,E) \), we view the following sets as \((0,1)\) vectors in \( \mathbb{R}^{|E|} \) in the standard way. Find an integer programming formulation and prove its correctness:

   (a) The family of Hamiltonian paths of \( G \) with endnodes \( u, v \). (A Hamiltonian path is a path that goes exactly once through each node/vertex of the graph.)
   (b) The family of all Hamiltonian paths of \( G \).
   (c) The family of edge sets that induce a triangle of \( G \).
   (d) Assuming that \( G \) has \( 3n \) nodes, the family of \( n \) node-disjoint triangles.
   (e) The family of odd cycles of \( G \).
Research question: For each problem above, is it possible to find a formulation using polynomially many inequalities (in the size of the graph \( G \)), or show that no such formulation exists?

[Problem 2.21 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]

9. (Playing with \((0,1)\)-vectors). Find integer programming formulations for the following integer sets.

(a) The set of all \((0,1)\)-vectors in \( \mathbb{R}^4 \) except \[
\begin{pmatrix}
0 \\
1 \\
1 \\
0
\end{pmatrix}.
\]

(b) The set of all \((0,1)\)-vectors in \( \mathbb{R}^6 \) except \[
\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.
\]

(c) The set of all \((0,1)\)-vectors in \( \mathbb{R}^6 \) except all the vectors having exactly two 1s in the first 3 components and one 1 in the last 3 components.

(d) The set of all \((0,1)\)-vectors in \( \mathbb{R}^n \) with an even number of 1s. You don’t have to find a system with \( \text{poly}(n) \) inequalities.

(e) The set of all \((0,1)\)-vectors in \( \mathbb{R}^n \) with an odd number of 1s. You don’t have to find a system with \( \text{poly}(n) \) inequalities.

Research question: For problems (d) and (e), is it possible to find a formulation using \( \text{poly}(n) \) many inequalities, or show that no such formulation exists?

[Adapted from Problem 2.27 from the “Integer Programming” textbook by Conforti, Cornuéjols and Zambelli.]