1. (Problem 6.36 in textbook) Consider the integer program for the stable set problem in a graph $G = (V, E)$:

$$\begin{align*}
\text{max} & \quad \sum_{v \in V} z_v \\
\text{subject to} & \quad z_v + z_w \leq 1 \quad \forall v, w \in V \text{ such that } vw \in E \\
& \quad z_v \geq 0 \quad \forall v \in V \\
& \quad z_v \in \mathbb{Z} \quad \forall v \in V
\end{align*}$$

Find cutting plane proofs using sequences of C-G cuts (starting from the above system) for the following “combinatorial” family of cutting planes:

(i) Let $C$ be an odd cycle. It is easy to see that the maximum number of vertices from $C$ in a stable set is at most $\frac{|C| - 1}{2}$. Thus, we have the family of “odd cycle inequalities”:

$$\sum_{v \in C} z_v \leq \frac{|C| - 1}{2} \quad \text{for all odd cycles } C.$$  

(ii) The following graph is known as a 5-wheel.

![5-wheel graph](image)

We have the “wheel inequalities”: Let $W$ be the node set of a 5-wheel with $r$ as the center node. Then the following is a valid inequality for the integer points $2z_r + \sum_{v \in W \setminus \{r\}} z_v \leq 2$. (Convince yourself of the validity of the inequality using combinatorial arguments first)

(iii) A set of vertices $K \subseteq V$ forms a clique if there is an edge between every pair of vertices in $K$. Clearly, we can have at most one vertex from any clique; thus, we have the family of “clique” inequalities $\sum_{v \in K} z_v \leq 1$ for all cliques $K$.

2. The knapsack problem consists of a set of $n$ items with weights $a_1, \ldots, a_n$. Each item $i \in \{1, \ldots, n\}$ has a value $c_i$. We have a knapsack that can hold a total weight of at most $W$. The goal is to find the subset of items with maximum total value that can fit into the knapsack.

One can formulate an integer program for the knapsack problem:

$$\begin{align*}
\text{max} & \quad c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \\
\text{subject to} & \quad a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \leq W \\
& \quad x_i \geq 0 \quad \forall i \in \{1, \ldots, n\} \\
& \quad x_i \leq 1 \quad \forall i \in \{1, \ldots, n\} \\
& \quad x_i \in \mathbb{Z} \quad \forall i \in \{1, \ldots, n\}
\end{align*}$$

We have the family of cutting planes known as “cover inequalities”. Let $C \subseteq \{1, \ldots, n\}$ be a subset such that $\sum_{i \in C} a_i > W$. Then $C$ is called a cover. $C$ is a minimal cover if $C \setminus \{j\}$ is not a cover for every $j \in C$. For every minimal cover $C$, a feasible solution can pick at most $|C| - 1$ items from the set $C$. Thus, the following inequality is valid: $\sum_{i \in C} x_i \leq |C| - 1$ for every minimal cover $C$. Show how to obtain the cover inequalities as CG cuts starting from the system above.
Chvátal-Gomory Closures. Suppose $P$ is a rational polytope, i.e., $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$ for some rational $A, b$. The minimum natural number $k$ such that $P^{(k)} = P_I$ is called the Chvátal closure rank of $P$.

(i) (Problem 6.32 from textbook) Let $k$ be a positive integer and let

$$\text{conv}(\{(0, 0), (1, 0), (\frac{1}{2}, k)\}) \subseteq \mathbb{R}^2.$$ 

Show that the Chvátal closure rank of $P$ is at least $k$. [Hint: Induction on $k$]

(ii) What is Chvátal rank of the odd set inequalities $\sum_{e \in E[S]} x_e \leq (|S| - 1)/2 \ (|S| \text{ odd})$ for the matching polytope?