AMS 550.666: Combinatorial Optimization
Homework Problems - Week VII

1. Let $M$ be an $m \times n$ totally unimodular matrix. Show that
\[
\begin{bmatrix}
M & 0 \\
I_n & I_n
\end{bmatrix}
\]
is totally unimodular.

2. Let $A$ be an $m \times n$ matrix such that every entry is either 0 or 1. Moreover, in any column the 1’s are consecutive, i.e., every column is of the following form:
\[
[0,0,\ldots,0,1,1,\ldots,1,0,0,\ldots,0]^T.
\]
Show that $A$ is totally unimodular.

3. (Problem 6.23 in textbook) Let $A$ be an $m \times n$ matrix with integer entries. Show that $A$ is totally unimodular if and only if every $m \times m$ submatrix of $[A \ I]$ has determinant $0, 1, -1$, where $I$ is the $m \times m$ identity matrix.

4. (Problem 6.24 in textbook) Let $A$ be a $n \times m$ totally unimodular matrix of full row rank and let $B$ be an $m \times m$ invertible submatrix of $A$. Show that $B^{-1}A$ is totally unimodular.
[Hint: Use the previous exercise]

5. For any undirected graph $G$, let $A_G$ denote its incidence matrix. Show that if $A_G$ is totally unimodular, then $G$ is bipartite.

6. Given an undirected graph, a clique is a subgraph that is complete, i.e. all pairs of vertices in the subgraph have an edge between them. In other words, a clique is given by a subset of the vertices that are all connected to each other by edges.

Write an integer program to find the largest clique (in terms of number of vertices) in a graph $G = (V, E)$. Let $G' = (V, E')$ be the graph on the same set of vertices $V$ such that $ij \in E'$ if and only if $ij \notin E$. Interpret the dual of the LP relaxation of the largest clique IP for $G$ in terms of $G'$. 