1. (Problem 6.36 in textbook) Consider the integer program for the stable set problem in a graph $G = (V, E)$:

\[
\begin{align*}
\text{max} & \quad \sum_{v \in V} z_v \\
\text{subject to} & \quad z_v + z_w \leq 1 \quad \forall v, w \in V \text{ such that } vw \in E \\
& \quad z_v \geq 0 \quad \forall v \in V \\
& \quad z_v \in \mathbb{Z} \quad \forall v \in V
\end{align*}
\]

Find cutting plane proofs using sequences of C-G cuts (starting from the above system) for the following “combinatorial” family of cutting planes:

(i) Let $C$ be an odd circuit. It is easy to see that the maximum number of vertices from $C$ in a stable set is at most $\frac{|C| - 1}{2}$. Thus, we have the family of “odd circuit inequalities”:

\[
\sum_{v \in C} z_v \leq \frac{|C| - 1}{2}
\]

for all odd circuits $C$.

Solution: Add the $|C|$ inequalities $z_v + z_w \leq 1$ when $vw \in C$ with multipliers $1/2$ to get $\sum_{v \in C} z_v \leq |C|/2$. Since $C$ is odd, $|C|/2 = \frac{|C| - 1}{2}$.

(ii) The following graph is known as a 5-wheel.

\[\text{We have the “wheel inequalities” : Let } W \text{ be the node set of a } 5 - \text{wheel with } r \text{ as the center node. Then the following is a valid inequality for the integer points } 2z_r + \sum_{v \in W \setminus \{r\}} z_v \leq 2. \text{ (Convince yourself of the validity of the inequality using combinatorial arguments first)}\]

Solution: For each triangle formed by $r$ and two other vertices of the cycle $v, w$ we have the odd circuit inequality $z_r + z_v + z_w \leq 1$. If we add these 5 triangle inequalities together, we get $5z_r + \sum_{v \in W \setminus \{r\}} 2z_v \leq 5$. Add the inequality $-z_r \leq 0$ to this to get $4z_r + \sum_{v \in W \setminus \{r\}} 2z_v \leq 5$. Divide through and round down the right hand side to get the desired inequality. Thus the 5-wheel inequality has Chvátal rank 2.

(iii) A set of vertices $K \subseteq V$ forms a clique if there is an edge between every pair of vertices in $K$. Clearly, we can have at most one vertex from any clique; thus, we have the family of “clique” inequalities $\sum_{v \in K} z_v \leq 1$ for all cliques $K$.

Solution: We prove by induction on the size of the clique. It is true when the clique is of size 2, i.e., it is an edge: this is simply an inequality in the original system. Suppose we have cutting plane proofs for all clique inequalities of size $n$. Consider a clique of size $n+1$ and remove one vertex $u$ from this clique to get a clique $K$ of size $n$. We have a cutting plane proof for $\sum_{v \in K} z_v \leq 1$ by the induction hypothesis. We add all the $n$ inequalities $z_u + z_w \leq 1$ for $w \in K$ and add this to $n - 1$ times the inequality $\sum_{v \in K} z_v \leq 1$. This gives $\sum_{v \in K \cup \{u\}}nz_v \leq 2n - 1$. Dividing through by $n$ and rounding down the right hand side gives the clique inequality for the clique $K \cup \{u\}$. This derivation shows the Chvátal rank 1.
2. The **knapsack problem** consists of a set of $n$ items with weights $a_1, \ldots, a_n$. Each item $i \in \{1, \ldots, n\}$ has a value $c_i$. We have a knapsack that can hold a total weight of at most $W$. The goal is to find the subset of items with maximum total value that can fit into the knapsack.

One can formulate an integer program for the knapsack problem:

$$\begin{align*}
\text{max} & \quad c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} & \quad a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq W \\
& \quad x_i \geq 0 \quad \forall i \in \{1, \ldots, n\} \\
& \quad x_i \leq 1 \quad \forall i \in \{1, \ldots, n\} \\
& \quad x_i \in \mathbb{Z} \quad \forall i \in \{1, \ldots, n\}
\end{align*}$$

We have the family of cutting planes known as “cover inequalities”. Let $C \subseteq \{1, \ldots, n\}$ be a subset such that $\sum_{i \in C} a_i > W$. Then $C$ is called a cover. $C$ is a minimal cover if $C \setminus \{j\}$ is not a cover for every $j \in C$. For every minimal cover $C$, a feasible solution can pick at most $|C| - 1$ items from the set $C$. Thus, the following inequality is valid: $\sum_{i \in C} x_i \leq |C| - 1$ for every minimal cover $C$. Show how to obtain the cover inequalities as CG cuts starting from the system above.

**Solution:** Let $C$ be a minimal cover. We now consider two cases:

**Case 1:** There exists $a_j$, $j \in C$ such that $a_j > W$. This means that $C = \{j\}$ since $C$ is a minimal cover. Then adding the inequalities $-x_i \leq 0$, $i \neq j$ with the multiplier $a_i$ to the knapsack inequality $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq W$ and then dividing through by $a_j$ we get the inequality $x_j \leq \frac{W}{a_j}$. Rounding down the right hand side gives $x_j \leq 0$ as desired.

**Case 2:** All $a_i \leq W$ for $i \in C$. For each $i \in C$ we give the inequality $x_i \leq 1$ the multiplier $\frac{W + 1 - a_i}{W + 1}$, we give the multiplier $\frac{a_i}{W + 1}$ for the inequality $-x_i \leq 0$ for each $i \notin C$, and we give the inequality $a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq W$ a multiplier $\frac{1}{W + 1}$. We add these inequalities together with these multipliers to get

$$\sum_{i \in C} \frac{W + 1 - a_i}{W + 1} x_i + \sum_{i \notin C} -\frac{a_i}{W + 1} x_i + \sum_{i = 1}^{n} \frac{a_i}{W + 1} x_i \leq |C| - \frac{1}{W + 1} \sum_{i \in C} a_i + \frac{W}{W + 1}$$

which simplifies to

$$\sum_{i \in C} \frac{W + 1}{W + 1} x_i \leq |C| + \frac{W - \sum_{i \in C} a_i}{W + 1}. \quad (1)$$

Since $C$ is a cover, $W - \sum_{i \in C} a_i < 0$. Since $C$ is minimal, $\sum_{i \in C} a_i - a_j \leq W$ for every $j \in C$. Thus, $\sum_{i \notin C} a_i - W \leq a_j \leq W$. Thus, $\sum_{i \notin C} \frac{a_i - W}{W + 1} < 1$. Thus, rounding down the right hand side of (1) gives $\sum_{i \in C} x_i \leq |C| - 1$.

3. Chvátal-Gomory Closures

(i) Suppose $P$ is a polytope. Show that there exists some natural number $k$ such that $P^{(k)} = P_I$. The minimum such $k$ is called the Chvátal closure rank of $P$.

**Solution:** By Problem 5 on HW VI, we know that $P_I$ is a polytope and hence can be described as the intersection of finitely many linear inequalities. Each of these inequalities have finite Chvátal rank by the theorem from class. Hence the Chvátal rank of $P_I$ is the maximum of the Chvátal ranks of the finitely many inequalities describing $P_I$.

(ii) (Problem 6.32 from textbook) Let $k$ be a positive integer and let

$$\text{conv}(\{(0,0), (0,1), \left(\frac{1}{2}, k\right)\}) \subseteq \mathbb{R}^2.$$
Show that the Chvátal closure rank of $P$ is at least $k$.

Solution: We prove it by induction. If $k = 1$, then since $(1/2, 1)$ is not an integer point, we have to use at least one C-G cut to shave off this point; thus, the Chvátal rank is at least 1. We assume the result holds for $k > 1$, and show that it holds for $k + 1$. Consider any C-G cut $w_1x_1 + w_2x_2 \leq \lfloor t \rfloor$ obtained by rounding down the right hand side of a valid inequality $w_1x_1 + w_2x_2 \leq t$ where $w_1, w_2 \in \mathbb{Z}$. Since this is valid inequality, it is valid for the points $\{(0,0), (1/2, (k+1))\}$; thus, $0 \leq t$ and $\frac{w_1}{2} + (k+1)w_2 \leq t$. We now consider two cases:

**Case 1:** $w_2 > 0$. Then since $w_2$ is an integer, $w_2 > t - \lfloor t \rfloor$. Therefore, $\lfloor t \rfloor > t - w_2 \geq \frac{w_1}{2} + (k+1)w_2 - w_2 = \frac{w_1}{2} + kw_2$. This shows that $(1/2, k)$ is valid for $w_1x_1 + w_2x_2 \leq \lfloor t \rfloor$.

**Case 2:** $w_2 \leq 0$. Since $(1,0)$ satisfies $w_1x_1 + w_2x_2 \leq t$, we have $w_1 \leq t$ and moreover, $w_1$ is an integer so $w_1 \leq \lfloor t \rfloor$. Further, since $0 \leq t$, we have $w_1/2 \leq \lfloor t \rfloor$. Since $w_2 \leq 0$, $kw_2 \leq 0$ since $k$ is a positive integer. Therefore, $w_1/2 + kw_2 \leq w_1/2 \leq \lfloor t \rfloor$.

Thus, the point $(1/2, k)$ satisfies all the rank 1 C-G cuts. Of course, $(0,0)$ and $(1,0)$ also satisfy all rank 1 C-G cuts. Therefore $\text{conv}(\{(0,0), (1/2, k)\}) \subseteq P^{(1)}$. By the induction hypothesis, $\text{conv}(\{(0,0), (0,1), (1/2, k)\})$ has Chvátal rank at least $k$. Therefore, the Chvátal rank of $P$ is at least $k + 1$.

(iii) What is Chvátal rank of the odd set inequalities $\sum_{e \in E[S]} x_e \leq (|S| - 1)/2$ (|S| odd) for the matching polytope?

Solution: The Chvátal rank of these odd set inequalities is 1. Add all the inequalities $\sum_{e \in \delta(v)} x_e \leq 1$ corresponding to $v \in S$ with multiplier $1/2$, together with the inequalities $-x_e \leq 0$ for all $e \in \delta(S)$ with multiplier $1/2$. This results in $\sum_{e \in E[S]} x_e \leq |S|/2$. Since $|S|$ is odd, rounding down the right hand side gives the odd-set inequality as desired.