AMS 550.666: Combinatorial Optimization
Homework Problems - Week VI

1. Let $X$ be a finite set in $\mathbb{R}^n$ and let $c \in \mathbb{R}^n$ be any vector. Show that

(i) All vertices of $\text{conv}(X)$ are contained in $X$.

Solution: Any element of $\text{conv}(X)$ is the convex combination of vectors in $X$. If we consider a vertex, then it is also a convex combination of vectors in $X$. But from the definition of a vertex, it cannot be the convex combination of DISTINCT points in $\text{conv}(X) \supseteq X$. Thus, the vertex must be the convex combination of a single vector in $X$, and hence it must be in $X$.

(ii) $\max\{c^T x : x \in X\} = \max\{c^T x : x \in \text{conv}(X)\}$.

Solution: Since $X \subseteq \text{conv}(X)$, the LHS is less than or equal to the RHS. From the result in class, there exists a vertex $\bar{x}$ of $\text{conv}(X)$ such that $\max\{c^T x : x \in \text{conv}(X)\} = c^T \bar{x}$. By part (i), $\bar{x} \in X$. Therefore, $\max\{c^T x : x \in X\} \geq c^T \bar{x} = \max\{c^T x : x \in \text{conv}(X)\}$. Thus, the LHS is greater than or equal to the RHS.

This result is the main observation which lets us potentially solve every combinatorial optimization problem using linear programming, provided that we can find a good inequality description for $\text{conv}(X)$ (once we have embedded the feasible solution of the combinatorial problem as the set of points $X \subseteq \mathbb{R}^n$.) Note that we did not really need the assumption of finiteness on $X$. If we consider more general $X$, the “max” must be replaced by a “sup” in part (ii), since $\text{conv}(X)$ may not be a polyhedron anymore. Everything else holds.

2. Let $P \subseteq \mathbb{R}^n$ be a polytope. Show that the convex hull of $P \cap \mathbb{Z}^n$ is a polytope. Is this true if $P$ is an unbounded polyhedron?

Solution: If $P$ is a polytope, it is a bounded set. This means that $P \cap \mathbb{Z}^n$ is finite and so by the Minkowski-Weyl theorem, the convex hull is a polytope. The result is not true if $P$ is an unbounded polyhedron: Let $P = \{(x_1, x_2) : x_2 \leq 0, -\sqrt{2}x_1 + x_2 \leq 0\}$. The convex hull of $P \cap \mathbb{Z}^n$ has infinitely many edges because the integer points keep getting closer and closer to the line $x_2 = \sqrt{2}x_1$. This forms a kind of “piecewise linear asymptote”.

3. Let $P \subseteq \mathbb{R}^n$ be a polytope. Show that $P = \text{conv}(P \cap \mathbb{Z}^n)$ if and only if all extreme points of $P$ are integral, i.e., have integer coordinates.

Solution: This follows from the facts that $P$ is the convex hull of its extreme points, and the extreme points of $\text{conv}(P \cap \mathbb{Z}^n)$ are integral by part (i) of Problem 1 above.

4. Show that the following integer program solves the minimum vertex cover problem in graph $G = (V, E)$. We have a variable $z_v$ associated with each vertex $v \in V$.

$$
\min \sum_{v \in V} z_v
\text{subject to} \quad z_v + z_w \geq 1 \quad \forall vw \in E
\quad z_v \geq 0 \quad \forall v \in V
\quad z_v \in \mathbb{Z} \quad \forall v \in V
$$

Solution: Notice that in any optimal solution, $z_v \leq 1$. Otherwise, we can strictly reduce its value down to 1; this keeps all the constraints satisfied, and strictly reduces the objective value. Thus, combined with the constraint $z_v \in \mathbb{Z}$ and $z_v \geq 0$, all optimal solutions are $0-1$ solutions. Notice that any such $0-1$ solution $z^*$ gives a vertex cover $\{v \in V : z^*_v = 1\}$. Conversely, any vertex cover $A$ gives a $0-1$ solution $z^*_v = 1$ if and only if $v \in A$. Therefore, the vertex covers for $G$ are in 1-1 correspondence with 0-1 solutions of the above integer program, and hence the optimal solution gives an optimal vertex cover.