1. Let $X$ be a finite set in $\mathbb{R}^n$ and let $c \in \mathbb{R}^n$ be any vector. Show that

(i) All vertices of $\text{conv}(X)$ are contained in $X$.

Solution: Any element of $\text{conv}(X)$ is the convex combination of vectors in $X$. If we consider a vertex, then it is also a convex combination of vectors in $X$. But from the definition of a vertex, it cannot be the convex combination of DISTINCT points in $\text{conv}(X) \supseteq X$. Thus, the vertex must be the convex combination of a single vector in $X$, and hence it must be in $X$.

(ii) $\max\{c^T x : x \in X\} = \max\{c^T x : x \in \text{conv}(X)\}$.

Solution: Since $X \subseteq \text{conv}(X)$, the LHS is less than or equal to the RHS. From the result in class, there exists a vertex $\bar{x}$ of $\text{conv}(X)$ such that $\max\{c^T x : x \in \text{conv}(X)\} = c^T \bar{x}$. By part (i), $\bar{x} \in X$. Therefore, $\max\{c^T x : x \in X\} \geq c^T \bar{x} = \max\{c^T x : x \in \text{conv}(X)\}$. Thus, the LHS is greater than or equal to the RHS.

This result is the main observation which lets us potentially solve every combinatorial optimization problem using linear programming, provided that we can find a good inequality description for $\text{conv}(X)$ (once we have embedded the feasible solution of the combinatorial problem as the set of points $X \subseteq \mathbb{R}^n$.) Note that we did not really need the assumption of finiteness on $X$. If we consider more general $X$, the “max” must be replaced by a “sup” in part (ii), since $\text{conv}(X)$ may not be a polyhedron anymore. Everything else holds.

2. Let $P \subseteq \mathbb{R}^n$ be a polytope. Show that the convex hull of $P \cap \mathbb{Z}^n$ is a polytope. Is this true if $P$ is an unbounded polyhedron?

Solution: If $P$ is a polytope, it is a bounded set. This means that $P \cap \mathbb{Z}^n$ is finite and so by the Minkowski-Weyl theorem, the convex hull is a polytope. The result is not true if $P$ is an unbounded polyhedron: Let $P = \{(x_1, x_2) : -x_2 \leq 0, -\sqrt{2}x_1 + x_2 \leq 0\}$. The convex hull of $P \cap \mathbb{Z}^n$ has infinitely many edges because the integer points keep getting closer and closer to the line $x_2 = \sqrt{2}x_1$. This forms a kind of “piecewise linear asymptote”.

3. Let $P \subseteq \mathbb{R}^n$ be a polytope. Show that $P = \text{conv}(P \cap \mathbb{Z}^n)$ if and only if all extreme points of $P$ are integral, i.e., have integer coordinates.

Solution: This follows from the facts that $P$ is the convex hull of its extreme points, and the extreme points of $\text{conv}(P \cap \mathbb{Z}^n)$ are integral by part (i) of Problem 1 above.

4. Let $M$ be an $m \times n$ totally unimodular matrix. Show that

$$
\begin{bmatrix}
M & 0 \\
I_n & I_n
\end{bmatrix}
$$

is totally unimodular.

Solution: Let the columns corresponding to $M$ in the above matrix be indexed by $j \in J = \{1, \ldots, n\}$ and let the remaining columns be indexed by $j' \in J' = \{1, \ldots, n\}$. Similarly, let the rows be partitioned into $I = \{1, \ldots, m\}$ and $I' = \{1, \ldots, n\}$, with $I$ corresponding to $M$. Consider any submatrix $B$ of

$$
\begin{bmatrix}
M & 0 \\
I_n & I_n
\end{bmatrix},
$$

with rows indexed by $I_1 \subseteq I$ and $I_1' \subseteq I'$, and columns indexed by $J_1 \subseteq J$ and $J_1' \subseteq J'$. 


If there exists \( j \in J_1' \) that is a column of \( B \), and \( j \notin I_1' \), then the column corresponding to \( j \) in \( B \) is a 0 column. Thus the determinant is 0. Thus, \( B \) can be expressed as

\[
\begin{bmatrix}
M' & 0 \\
X & I_k
\end{bmatrix},
\]

where \( k = |J_1'| \) and \( M' \) is a square submatrix of \( \begin{bmatrix} M & I \end{bmatrix} \). Since \( \begin{bmatrix} M & I \end{bmatrix} \) is totally unimodular, this shows that determinant of \( B = \det(M') \in \{0, 1, -1\} \).

5. (Problem 6.23 in textbook) Let \( A \) be an \( m \times n \) matrix with integer entries. Show that \( A \) is totally unimodular if and only if every \( m \times m \) submatrix of \( [A \ I] \) has determinant 0, \( 1 \), or \(-1\), where \( I \) is the \( m \times m \) identity matrix.

Solution: Every \( m \times m \) submatrix of \( [A \ I] \) is of the form

\[
\begin{bmatrix}
A' & 0 \\
X & I_k
\end{bmatrix},
\]

for some natural number \( k \leq m \) and \( A' \) a square submatrix of \( A \). Since the determinant of this \( m \times m \) submatrix equals \( \det(A') \), the assertion follows immediately.

6. (Problem 6.24 in textbook) Let \( A \) be a totally unimodular matrix of full row rank and let \( B \) be an \( m \times m \) invertible submatrix of \( A \). Show that \( B^{-1}A \) is totally unimodular. [Hint: Use the previous exercise]

Solution:

**Claim 1.** Every \( m \times m \) submatrix of \( B^{-1}A \) has determinant 0, 1 or \(-1\).

**Proof.** Any sub matrix is of the form \( B^{-1}A' \) where \( A' \) is an \( m \times m \) submatrix of \( A \). Therefore the determinant of equals \( \det(A')/\det(B) \) and since \( A' \) and \( B \) are both submatrices of \( A \) which is totally unimodular, we have the claim. \( \square \)

We can write \( A = [B \ N] \) and therefore \( B^{-1}A = [I \ B^{-1}N] \). Since by the claim, every \( m \times m \) submatrix of \( [I \ B^{-1}N] \) has determinant 0, 1 or \(-1\), Problem 6 tells us that \( B^{-1}N \) is totally unimodular. Therefore, \( [I \ B^{-1}N] = B^{-1}A \) is totally unimodular.

7. For any graph \( G \), let \( A_G \) denote its incidence matrix. Show that if \( A_G \) is totally unimodular, then \( G \) is bipartite. [We proved the converse in class]

Solution: We prove the contrapositive. Suppose \( G \) is not bipartite. Then there exists an odd cycle \( C \). Consider the submatrix of \( A_G \) with rows indexed by the vertices of \( C \) and the columns are indexed by the edges of \( C \). The determinant of this matrix is 2, showing that \( A_G \) is not TU.

8. (Birkhoff’s permutation polytope theorem) An \( n \times n \) matrix is called *doubly stochastic* if every entry is a nonnegative real number and the sum of the entries in each row and each column is exactly 1. Any permutation \( \sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \) gives a permutation matrix \( P^\sigma \) which is a \( \{0, 1\} \ n \times n \) matrix such that \( P^\sigma_{ij} = 1 \) if and only if \( \sigma(i) = j \). Show that for any doubly stochastic matrix \( M \), there exist finitely many permutation matrices \( P^{\sigma_1}, \ldots, P^{\sigma_m} \) and convexity coefficients \( \lambda_1, \ldots, \lambda_m \geq 0 \) with \( \sum_{k=1}^{m} \lambda_k = 1 \) such that \( M = \lambda_1 P^{\sigma_1} + \ldots + \lambda_m P^{\sigma_m} \).

Solution: Let \( x_{ij} \) denote the \( ij \)-th entry of a doubly stochastic matrix \( M \). Then we have for any row \( i, \sum_{j=1}^{n} x_{ij} = 1 \) and for any column \( j, \sum_{i=1}^{n} x_{ij} = 1 \). Of course, by definition,
\( x_{ij} \geq 0 \). Now consider a bipartite graph given by vertices \( V_1 \cup V_2 \) with \( |V_1| = |V_2| = n \). We know that the matching polytope for bipartite graphs is given by

\[
\sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \\
x_e \geq 0 \quad \forall e \in E
\]

which in this case simplifies to

\[
\sum_{j=1}^{n} x_{ij} \leq 1 \quad \forall i \in \{1, \ldots, n\} \\
\sum_{i=1}^{n} x_{ij} \leq 1 \quad \forall j \in \{1, \ldots, n\} \\
x_{ij} \geq 0 \quad \forall i, j
\]

Thus the entries of a doubly stochastic matrices satisfy the above constraints. Recall that all the points in the above polytope are convex hulls of the vertices and the vertices are integral (corresponding to the matchings in \( G \)). So any \( x \) corresponding to a doubly stochastic matrix is the convex hull of vertices \( v^1, \ldots, v^m \). Since for any row \( i \), \( \sum_{j=1}^{n} x_{ij} = 1 \) and for any column \( j \), \( \sum_{i=1}^{n} x_{ij} = 1 \), the same equalities hold for \( v^1, \ldots, v^m \), i.e., for every \( k \in \{1, \ldots, m\} \) we have that \( \sum_{j=1}^{n} v^k_{ij} = 1 \) for any \( i \) and \( \sum_{i=1}^{n} x_{ij} = 1 \) for any \( j \). Finally observe that an integral solution satisfying these equalities (thus, corresponding to a perfect matching in \( G \)) corresponds to a permutation matrix and thus we are done.