AMS 553.766: Combinatorial Optimization

Homework Problems - Week V

For the following problems, \( A \in \mathbb{R}^{m \times n} \) will be \( m \times n \) matrices, and \( b \in \mathbb{R}^m \). An affine subspace is the set of solutions to a a system of linear equations, i.e., \( \{ x \in \mathbb{R}^n : Ax = b \} \) is an affine subspace of \( \mathbb{R}^n \). A polytope is a bounded polyhedron.

1. Show the following:
   
   (i) The intersection of an affine subspace with a polyhedron is a polyhedron.
   
   (ii) Let \( P_1, P_2 \) be two polytopes. Define \( C = \{ x + y : x \in P_1, y \in P_2 \} \). Show that \( C \) is a polytope.
   
   (iii) The intersection of two polyhedra is a polyhedron. Thus, show that the intersection of a polyhedron with a polytope is a polytope.
   
   (iv) Let \( T : \mathbb{R}^n \to \mathbb{R}^d \) be a linear map. Show that if \( P \subseteq \mathbb{R}^n \) is a polytope, then \( T(P) \) is a polytope in \( \mathbb{R}^d \).

   [You may quote the Minkowski-Weyl theorem without proof.]

2. (Problems 2.15, 2.16, 2.17 from Lex Schrijver’s notes) Prove the following Farkas’ type results.

   (i) Prove that there is no solution to \( Ax \leq b \) if and only if there exists \( y \geq 0 \) such that \( y^T A = 0 \) and \( y^T b < 0 \).
   
   (ii) Prove that there exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if for each \( y \geq 0, y^T A \geq 0 \Rightarrow y^T b \geq 0 \).
   
   (iii) Prove that there exists \( x > 0 \) satisfying \( Ax = 0 \) if and only if for each \( y \in \mathbb{R}^m, y^T A \geq 0 \Rightarrow y^T A = 0 \).
   
   (iv) Prove that there exists \( x \neq 0 \) satisfying \( x \geq 0 \) and \( Ax = 0 \) if and only if there is no vector \( y \in \mathbb{R}^m \) satisfying \( y^T A > 0 \).

3. (Complementary slackness) Let \( x^* \in \mathbb{R}^n \) be an optimal solution to the problem \( \max \{ c^T x : Ax \leq b \} \) and \( y^* \in \mathbb{R}^m \) be an optimal solution to the problem \( \max \{ y^T b : y^T A = c^T, y \geq 0 \} \).

   Show that for every \( i = 1, \ldots, m \) either \( a_i \cdot x^* = b_i \) or \( y_i^* = 0 \) (or both). (Here \( a_i \) denotes the \( i \)-th row of \( A \) and \( b_i \) is the \( i \)-th component of \( b \).) [Hint: consider the vector \( (y^*)^T (Ax^* - b) \)]

   The theorem is saying that in an optimal primal-dual pair of solutions for an LP, either a constraint is tight at the optimal primal solution or the corresponding dual multiplier is 0.

4. Let \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) be a nonempty polyhedron. Let \( C = \{ r \in \mathbb{R}^n : x + \lambda r \in P \text{ for all } x \in P, \lambda \in \mathbb{R}_+ \} \) where \( \mathbb{R}_+ \) is the set of nonnegative real numbers. Show that

   (i) \( C \) is a convex cone, i.e., for any \( r^1, r^2 \in C \) and \( \lambda_1, \lambda_2 \geq 0 \), we have \( \lambda_1 r^1 + \lambda_2 r^2 \in C \).

   [This cone is called the recession cone of the polyhedron \( P \)]
   
   (ii) \( C = \{ r \in \mathbb{R}^n : Ar \leq 0 \} \).

5. Let \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) be a nonempty polyhedron. Let \( L = \{ r \in \mathbb{R}^n : x + \lambda r \in P \text{ for all } x \in P, \lambda \in \mathbb{R} \} \). Show that

   (i) \( L \) is a linear subspace of \( \mathbb{R}^n \). [This is called the lineality space of the polyhedron \( P \)]
   
   (ii) \( L = \{ r \in \mathbb{R}^n : Ar = 0 \} \).

6. Using the notation from Problems 5 and 6, show that \( L = C \cap (-C) \). We say that \( P \) is pointed if \( L = \{ 0 \} \). Suppose \( P \neq \emptyset \); show that \( P \) has at least one vertex if and only if \( P \) is pointed.
7. (The Diet Problem) Suppose you want to design a diet for your meals. You have certain food items (e.g., spinach, chicken, rice etc.); let us label these different food types as $f_1, \ldots, f_n$. You can choose any nonnegative amount of a food item to put in your diet. Each food item has a per unit cost $c_1, \ldots, c_n$ associated with it. You have to meet some nutritional constraints: for example, you must have at least 5g of protein, and at most 40g of protein in the meal. Let us say there are $\{1, \ldots, k\}$ nutritional categories and each category has a lower bound $\ell_i$ and an upper bound $u_i$ that must be met. Suppose that each unit of food item $f_j$ provides $a_{ij}$ units of nutritional category $i$. How will you solve the problem of designing a diet satisfying the nutritional demands that has the least cost?

8. (Linear Regression with different objectives) In linear regression, we have a bunch of labeled data points $z^1, \ldots, z^k \in \mathbb{R}^n$ with real valued labels $y_1, \ldots, y_k$. We want to fit the best linear function to this labeled data. More precisely, we want to find parameters $\beta = (\beta_1, \ldots, \beta_n)$ so as to minimize the errors $|y_j - \sum_{i=1}^n \beta_i z_{ji}|$. The typical objective is the sum of the squares of the errors, i.e., we wish to minimize $\sum_{j=1}^k (y_j - \sum_{i=1}^n \beta_i z_{ji})^2$. Suppose we are interested in the following variant:

Firstly, we don’t want to allow arbitrary values of the parameters; we want more controlled regression. Suppose for each parameter $\beta_i$, we have certain preset upper and lower bounds $u_i$ and $\ell_i$ respectively that we want the parameter to lie within. Also, instead of minimizing the sum of squares, suppose we want to minimize the sum of the absolute values, i.e., minimize $\sum_{j=1}^k |y_j - \sum_{i=1}^n \beta_i z_{ji}|$ subject to these bound constraints on the parameter values ($\ell_1$ minimization).

Show how you can formulate this as a linear programming problem. What if you were interested in minimizing the largest error, i.e., minimize $\max_{j \in \{1, \ldots, k\}} |y_j - \sum_{i=1}^n \beta_i z_{ji}|$ ($\ell_\infty$ minimization)