AMS 550.666: Combinatorial Optimization
Homework Problems - Week V

For the following problems, $A \in \mathbb{R}^{m \times n}$ will be $m \times n$ matrices, and $b \in \mathbb{R}^m$. **An affine subspace** is the set of solutions to a system of linear equations, i.e., $\{x \in \mathbb{R}^n : Ax = b\}$ is an affine subspace of $\mathbb{R}^n$. A **polytope** is a bounded polyhedron.

1. Show the following:
   (i) The intersection of an affine subspace with a polyhedron is a polyhedron.
   (ii) Let $P_1, P_2$ be two polytopes. Define $C = \{x + y : x \in P_1, y \in P_2\}$. Show that $C$ is a polytope.
   (iii) The intersection of two polyhedra is a polyhedron. Thus, show that the intersection of a polyhedron with a polytope is a polytope.
   (iv) Let $T : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map. Show that if $P \subseteq \mathbb{R}^n$ is a polytope, then $T(P)$ is a polytope in $\mathbb{R}^d$.

   [You may quote the Minkowski-Weyl theorem without proof.]

2. Show that
   $$\max\{c^T x : Ax \leq b, x \geq 0\} = \min\{y^T b : y^T A \geq c^T, y \geq 0\}$$
   provided that both values are finite. [Adapt the standard LP duality result from class to incorporate the constraints $x \geq 0$ and then show that the right hand side above is equivalent to the dual LP]

3. (Complementary slackness) Let $x^* \in \mathbb{R}^n$ be an optimal solution to the problem $\max\{c^T x : Ax \leq b\}$ and $y^* \in \mathbb{R}^m$ be an optimal solution to the problem $\max\{y^T b : y^T A = c^T, y \geq 0\}$. Show that for every $i = 1, \ldots, m$ either $a_i \cdot x^* = b_i$ or $y^*_i = 0$ (or both). (Here $a_i$ denotes the $i$-th row of $A$ and $b_i$ is the $i$-th component of $b$.) [Hint: consider the vector $(y^*)^T (Ax^* - b)$]
   The theorem is saying that in an optimal primal-dual pair of solutions for an LP, either a constraint is tight at the optimal primal solution or the corresponding dual multiplier is 0.

4. Find an example of a pair of primal and dual linear programs such that both problems are infeasible. [Hint: There is an example with 2 variables in each problem]

5. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a nonempty polyhedron. Let $C = \{r \in \mathbb{R}^n : x + \lambda r \in P \text{ for all } x \in P, \lambda \in \mathbb{R}_+\}$ where $\mathbb{R}_+$ is the set of nonnegative real numbers. Show that
   (i) $C$ is a convex cone. [This cone is called the recession cone of the polyhedron $P$]
   (ii) $C = \{r \in \mathbb{R}^n : Ar \leq 0\}$.

6. Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a nonempty polyhedron. Let $L = \{r \in \mathbb{R}^n : x + \lambda r \in P \text{ for all } x \in P, \lambda \in \mathbb{R}\}$. Show that
   (i) $L$ is a linear subspace of $\mathbb{R}^n$. [This is called the lineality space of the polyhedron $P$]
   (ii) $L = \{r \in \mathbb{R}^n : Ar = 0\}$.

7. Using the notation from Problems 5 and 6, show that $L = C \cap (-C)$. We say that $P$ is **pointed** if $L = \{0\}$. Suppose $P \neq \emptyset$; show that $P$ has at least one vertex if and only if $P$ is pointed.