AMS 550.666: Combinatorial Optimization
Homework Problems - Week VI

For the following problems, $A, M \in \mathbb{R}^{m \times n}$ will be $m \times n$ matrices, and $b \in \mathbb{R}^n$. An affine subspace is the set of solutions to a system of linear equations, i.e., $\{ x \in \mathbb{R}^n : Ax = b \}$ is an affine subspace of $\mathbb{R}^n$.

1. Show the following:

(i) The intersection of an affine subspace with a polyhedron is a polyhedron.

Solution: Express the original polyhedron as $P = \{ x : Ax \leq b \}$ and the affine subspace as $L = \{ x : A'x = b' \}$. The intersection $P \cap L = \{ x : A'x = b', Ax \leq b \} = \{ x : A'x \leq b', -A'x \leq -b', Ax \leq b \}$ which is by definition a polyhedron.

(ii) Let $P_1, P_2$ be two polytopes. Define $C = \{ x + y : x \in P_1, y \in P_2 \}$. Show that $C$ is a polytope.

Solution: Express $P_1 = \text{conv}\{x_1, \ldots, x_p\}$ and $P_2 = \text{conv}\{y_1, \ldots, y_q\}$. We claim that $C = \text{conv}\{x_i + y_j : i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}\}$. Consider $z \in C$, then $z = x + y$ for some $x \in P_1$ and $y \in P_2$. We can write $x = \sum_{i=1}^{p} \lambda_i x_i$ and $y = \sum_{j=1}^{q} \mu_j y_j$, where $\lambda_i \geq 0$, $\sum_{i=1}^{p} \lambda_i = 1$, $\mu_j \geq 0$ and $\sum_{j=1}^{q} \mu_j = 1$. Observe that $\sum_{i,j} \lambda_i \mu_j = 1$. Also,

$$\sum_{i,j} \lambda_i \mu_j (x_i + y_j) = \sum_{i=1}^{p} \lambda_i x_i + \sum_{j=1}^{q} \mu_j y_j = x + y = z.$$

Since $\sum_{i,j} \lambda_i \mu_j (x_i + y_j) \in \text{conv}\{x_i + y_j : i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}\}$, we have $z \in \text{conv}\{x_i + y_j : i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}\}$. Therefore, $C \subseteq \text{conv}\{x_i + y_j : i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}\}$. To show the reverse inclusion, consider $z \in \text{conv}\{x_i + y_j : i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}\}$; so we can write $z = \sum_{i,j} \sigma_{ij} (x_i + y_j)$ where $\sigma_{ij} \geq 0$ and $\sum_{i,j} \sigma_{ij} = 1$. We now simply this expression:

$$\sum_{i,j} \sigma_{ij} (x_i + y_j) = \sum_{i,j} \sigma_{ij} x_i + \sum_{i,j} \sigma_{ij} y_j = \left( \sum_{i=1}^{p} \left( \sum_{j=1}^{q} \sigma_{ij} \right) x_i \right) + \left( \sum_{j=1}^{q} \left( \sum_{i=1}^{p} \sigma_{ij} \right) y_j \right) = \sum_{i=1}^{p} \lambda_i x_i + \sum_{j=1}^{q} \mu_j y_j,$$

where $\lambda_i = \sum_{j=1}^{q} \sigma_{ij}$ and $\mu_j = \sum_{i=1}^{p} \sigma_{ij}$. Observe that

$$\sum_{i=1}^{p} \lambda_i = \sum_{j=1}^{q} \sum_{i=1}^{p} \sigma_{ij} = 1.$$

Similarly, $\sum_{j=1}^{q} \mu_j = 1$. Thus, $\sum_{i=1}^{p} \lambda_i x_i \in P_1$ and $\sum_{j=1}^{q} \mu_j y_j \in P_2$. $z = \sum_{i=1}^{p} \lambda_i x_i + \sum_{j=1}^{q} \mu_j y_j \in P_1 + P_2$.

Thus, we express $C$ as the convex hull of a finite set of points; by the Minkowski-Weyl theorem it is a polytope.

(iii) The intersection of two polyhedra is a polyhedron. Thus, show that the intersection of a polyhedron with a polytope is a polytope.

Solution: Let one polyhedron be represented by $A_1 x \leq b_1$ and the second polyhedron be $A_2 x \leq b_2$. The intersection is given by $A_1 x \leq b_1, A_2 \leq b_2$ which is another polyhedron. Since the intersection of a bounded set with another set is also bounded, the intersection of a polytope with another polyhedron is a polytope.
2. Show that

\[ \text{max} \{ c^T x : Ax \leq b, \; x \geq 0 \} = \min \{ y^T b : y^T A \geq c^T, \; y \geq 0 \} \]

provided that both values are finite. [Adapt the standard LP duality result from class to incorporate the constraints \( x \geq 0 \) and then show that the right hand side above is equivalent to the dual LP]

Solution: The primal problem is \( \text{max} \{ c^T x : A' x \leq b' \} \), where \( A' = \begin{bmatrix} A \\ I \end{bmatrix} \) and \( b' = \begin{bmatrix} b \\ 0 \end{bmatrix} \). By LP duality, \( \text{max} \{ c^T x : A' x \leq b' \} = \min_{y' \in \mathbb{R}^{m+n}} \{ y'^T b' : y'^T A' = c^T, \; y' \geq 0 \} \). Decompose \( y' \in \mathbb{R}^{m+n} \) as \( y' = \begin{bmatrix} y \\ s \end{bmatrix} \) where \( y \in \mathbb{R}^m \) and \( s \in \mathbb{R}^n \). Rewriting the dual, we get \( \min_{y' \in \mathbb{R}^{m+n}} \{ y'^T b' : y'^T A' = c^T, \; y' \geq 0 \} = \min_{y \in \mathbb{R}^m, s \in \mathbb{R}^n} \{ y^T b + s^T 0 : y^T A + s = c^T, \; y \geq 0, s \geq 0 \} \). The constraint \( s \geq 0 \) implies that \( \min_{y \in \mathbb{R}^m, s \in \mathbb{R}^n} \{ y^T b + s^T 0 : y^T A + s = c^T, \; y \geq 0, s \geq 0 \} = \min \{ y^T b : y^T A \geq c^T, \; y \geq 0 \} \). Thus, we are done.

3. (Complementary slackness) Let \( x^* \in \mathbb{R}^n \) be an optimal solution to the problem \( \text{max} \{ c^T x : Ax \leq b \} \) and \( y^* \in \mathbb{R}^m \) be an optimal solution to the problem \( \text{max} \{ y^T b : y^T A = c^T, \; y \geq 0 \} \).

Show that for every \( i = 1 \ldots, m \) either \( a_i \cdot x^* = b_i \) or \( y_i^* = 0 \) (or both). (Here \( a_i \) denotes the \( i \)-th row of \( A \) and \( b_i \) is the \( i \)-th component of \( b \).) [Hint: consider the vector \( (y^*)^T (Ax^* - b) \)]

The theorem is saying that in an optimal primal-dual pair of solutions for an LP, either a constraint is tight at the optimal primal solution or the corresponding dual multiplier is 0.

Solution: We will in fact show that two feasible solutions \( x^* \) (for the primal) and \( y^* \) (for the dual) are optimal solutions if and only if for every \( i = 1 \ldots, m \) either \( a_i \cdot x^* = b_i \) or \( y_i^* = 0 \) (or both). Observe that since \( x^* \) and \( y^* \) are feasible for the primal and dual problems respectively,

\[ c^T x^* - (y^*)^T b = ((y^*)^T A)x^* - (y^*)^T b = (y^*)^T (Ax^* - b) \leq 0 \]

Moreover, since each component of \( y^* \) is nonnegative and each component of \( Ax^* - b \) is also nonnegative, \( c^T x^* - (y^*)^T b = 0 \) if and only if for every \( i = 1 \ldots, m \) either \( a_i \cdot x^* = b_i \) or \( y_i^* = 0 \) (or both). Finally, \( c^T x^* - (y^*)^T b = 0 \) is equivalent to \( x^* \) and \( y^* \) being optimal solutions to the primal and dual problems respectively.

4. Let \( X \) be a finite set in \( \mathbb{R}^n \) and let \( c \in \mathbb{R}^n \) be any vector. Show that

(i) All vertices of \( \text{conv}(X) \) are contained in \( X \).

Solution: Any element of \( \text{conv}(X) \) is the convex combination of vectors in \( X \). If we consider a vertex, then it is also a convex combination of vectors in \( X \). But from the definition of a vertex, it cannot be the convex combination of DISTINCT points in
conv(X) ⊆ X. Thus, the vertex must be the convex combination of a single vector in
X, and hence it must be in X.

(ii) max\{c^T x : x ∈ X\} = max\{c^T x : x ∈ conv(X)\}.

Solution: Since X ⊆ conv(X), the LHS is less than or equal to the RHS. From the result
in class, there exists a vertex \(\bar{x}\) of conv(X) such that max\{c^T x : x ∈ conv(X)\} = c^T \bar{x}.
By part (i), \(\bar{x} ∈ X\). Therefore, max\{c^T x : x ∈ X\} ≥ c^T \bar{x} = max\{c^T x : x ∈ conv(X)\}.
Thus, the LHS is greater than or equal to the RHS.

This result is the main observation which lets us potentially solve every combinatorial op-
mimization problem using linear programming, provided that we can find a good inequality
description for conv(X) (once we have embedded the feasible solution of the combinatorial
problem as the set of points \(X ⊆ \mathbb{R}^n\).)

5. Let \(P ⊆ \mathbb{R}^n\) be a polytope. Show that the convex hull of \(P ∩ \mathbb{Z}^n\) is a polytope. Is this true if
\(P\) is an unbounded polyhedron?

Solution: If \(P\) is a polytope, it is a bounded set. This means that \(P ∩ \mathbb{Z}^n\) is finite and so by
the Minkowski-Weyl theorem, the convex hull is a polytope. The result is not true if \(P\) is an
unbounded polyhedron: Let \(P = \{(x_1, x_2) : -x_2 ≤ 0, -\sqrt{2} x_1 + x_2 ≤ 0\}\). The convex hull of
\(P ∩ \mathbb{Z}^n\) has infinitely many edges because the integer points keep getting closer and closer to
the line \(x_2 = \sqrt{2} x_1\). This forms a kind of “piecewise linear asymptote”.

6. Let \(M\) be an \(m × n\) totally unimodular matrix. Show that
\[
\begin{bmatrix}
M & 0 \\
I_n & I_n
\end{bmatrix}
\]
is totally unimodular.

Solution: Let the columns corresponding to \(M\) in the above matrix be indexed by \(j ∈ J = \{1, \ldots, n\}\) and let the remaining columns be indexed by \(j′ ∈ J′ = \{1, \ldots, n\}\). Similarly, let
the rows be partitioned into \(I = \{1, \ldots, m\}\) and \(I′ = \{1, \ldots, n\}\), with \(I\) corresponding to \(M\).
Consider any submatrix \(B\) of
\[
\begin{bmatrix}
M & 0 \\
I_n & I_n
\end{bmatrix},
\]
with rows indexed by \(I_1 ⊆ I\) and \(I_1′ ⊆ I′\), and columns indexed by \(J_1 ⊆ J\) and \(J_1′ ⊆ J′\).
If there exists \(j ∈ J_1′\) that is a column of \(B\), and \(j ∉ I_1\), then the column corresponding to \(j\)
in \(B\) is a 0 column. Thus the determinant is 0. Thus, \(B\) can be expressed as
\[
\begin{bmatrix}
M′ & 0 \\
X & I_k
\end{bmatrix},
\]
where \(k = |J_1′|\) and \(M′\) is a square submatrix of \(\begin{bmatrix} M \end{bmatrix}_I\). Since \(\begin{bmatrix} M \end{bmatrix}_I\) is totally unimodular,
this shows that determinant of \(B = det(M′) ∈ \{0, 1, -1\}\).

7. (Problem 6.23 in textbook) Let \(A\) be an \(m × n\) matrix with integer entries. Show that \(A\) is
totally unimodular if and only if every \(m × m\) submatrix of \([A \ I]\) has determinant 0, 1, \(-1,\)
where \(I\) is the \(m × m\) identity matrix.

Solution: Every \(m × m\) submatrix of \([A \ I]\) is of the form
\[
\begin{bmatrix}
A′ & 0 \\
X & I_k
\end{bmatrix},
\]
for some natural number \(k ≤ m\) and \(A′\) a square submatrix of \(A\). Since the determinant of
this \(m × m\) submatrix equals \(det(A′)\), the assertion follows immediately.
8. (Problem 6.24 in textbook) Let $A$ be a totally unimodular matrix of full row rank and let $B$ be an $m \times m$ invertible submatrix of $A$. Show that $B^{-1}A$ is totally unimodular. [Hint: Use the previous exercise]

Solution:

**Claim 1.** Every $m \times m$ submatrix of $B^{-1}A$ has determinant 0, 1 or $-1$.

*Proof.* Any sub submatrix is of the form $B^{-1}A'$ where $A'$ is an $m \times m$ submatrix of $A$. Therefore the determinant of equals $\det(A')/\det(B)$ and since $A'$ and $B$ are both submatrices of $A$ which is totally unimodular, we have the claim. \(\square\)

We can write $A = [B \ N]$ and therefore $B^{-1}A = [I \ B^{-1}N]$. Since by the claim, every $m \times m$ submatrix of $[I \ B^{-1}N]$ has determinant 0, 1 or $-1$, Problem 6 tells us that $B^{-1}N$ is totally unimodular. Therefore, $[I \ B^{-1}N] = B^{-1}A$ is totally unimodular.

9. For any graph $G$, let $A_G$ denote its incidence matrix. Show that if $A_G$ is totally unimodular, then $G$ is bipartite.

Solution: We prove the contrapositive. Suppose $G$ is not bipartite. Then there exists an odd cycle $C$. Consider the submatrix of $A_G$ with rows indexed by the vertices of $C$ and the columns are indexed by the edges of $C$. The determinant of this matrix is 2, showing that $A_G$ is not TU.