AMS 550.666: Combinatorial Optimization
Homework Problems - Week IV

For any two vectors \( x, y \in \mathbb{R}^d \), \( x \leq y \) means the inequality holds for each component: \( x_i \leq y_i \) for all \( i = 1, \ldots, d \). Similarly, \( x < y \) means \( x_i < y_i \) for all \( i = 1, \ldots, d \).

For the following problems, \( A \in \mathbb{R}^{m \times n} \) is an \( m \times n \) matrix, and \( b \in \mathbb{R}^m \).

1. Let \( X \) be an arbitrary (possibly infinite) subset of \( \mathbb{R}^n \). The convex hull of \( X \), denoted by \( \text{conv}(X) \), is a convex set \( C \) such that \( X \subseteq C \) and for any other convex set \( C' \), \( X \subseteq C' \Rightarrow C \subseteq C' \), i.e., the convex hull is the smallest (with respect to set inclusion) convex set containing \( X \). Show that

\[
\text{conv}(X) = \bigcap \{ C : X \subseteq C, \text{C convex} \} = \{ \lambda_1 x_1 + \ldots + \lambda_t x_t : x_1, \ldots, x_t \in X, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1 \}
\]

Solution: Let \( \hat{C} = \bigcap \{ C : X \subseteq C, \text{C convex} \} \). Consider any other convex set \( C' \) such that \( X \subseteq C' \). Then \( C' \) appears in the intersection, and thus \( \hat{C} \subseteq C' \). Thus, \( \hat{C} = \text{conv}(X) \).

Next, let \( \hat{C} = \{ \lambda_1 x_1 + \ldots + \lambda_t x_t : x_1, \ldots, x_t \in X, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1 \} \). Then,

(a) \( \hat{C} \) is convex. Consider two points \( z_1, z_2 \in \hat{C} \). Thus there exist two finite index sets \( I_1, I_2 \), two finite subsets of \( X \) given by \( X_1 = \{ x^i_1 \in X : i \in I_1 \} \) and \( X_2 = \{ x^i_2 \in X : i \in I_2 \} \), and two subsets of nonnegative real numbers \( \{ \lambda_1^i \geq 0, i \in I_1 \}, \{ \lambda_2^i \geq 0, i \in I_2 \} \) such that \( \sum_{i \in I_j} \lambda_j^i = 1 \) for \( j = 1, 2 \), with the following property: \( z_j = \sum_{i \in I_j} \lambda_j^i x^i_j \) for \( j = 1, 2 \),

Then for any \( \lambda \in [0, 1] \), \( \lambda z_1 + (1 - \lambda) z_2 = \lambda (\sum_{i \in I_1} \lambda_1^i x^i_1) + (1 - \lambda) (\sum_{i \in I_2} \lambda_2^i x^i_2) \). Consider the finite set \( \tilde{X} = X_1 \cup X_2 \), and for each \( x \in \tilde{X} \), if \( x = x_i \in X_1 \) with \( i \in I_1 \) let \( \mu_x = \lambda \cdot \lambda_1^i \), and if \( x = x_i \in X_2 \) with \( i \in I_2 \), let \( \mu_x = (1 - \lambda) \cdot \lambda_2^i \). It is easy to check that \( \sum_{x \in \tilde{X}} \mu_x x \) is convex, every point of the form \( \lambda_1 x_1 + \ldots + \lambda_t x_t \) where \( x_1, \ldots, x_t \in X_1, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1 \) belongs to \( C' \). Thus, \( \hat{C} \subseteq C' \).

(b) \( X \subseteq \hat{C} \). We simply use \( \lambda = 1 \) as the multiplier for a point from \( X \).

(c) Let \( C' \) be any convex set such that \( X \subseteq C' \). Since \( C' \) is convex, every point of the form \( \lambda_1 x_1 + \ldots + \lambda_t x_t \) where \( x_1, \ldots, x_t \in X_1, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1 \) belongs to \( C' \). Thus, \( \hat{C} \subseteq C' \).

From (a), (b) and (c), we get that \( \hat{C} = \text{conv}(X) \).

2. (Problem 2.2 in Schrijver’s notes) Let \( C \) be a convex set in \( \mathbb{R}^n \), and let \( A \) be any \( m \times n \) matrix. Show that the set \{ \( Ax : x \in C \) \} is convex. [Thus, convexity is preserved under linear transformations.] Let \( C' \) be a convex set in \( \mathbb{R}^m \). Show that \{ \( x \in \mathbb{R}^n : Ax \in C' \) \} is also a convex set.

Solution: Let \( D = \{ Ax : x \in C \} \). Consider \( Ax_1, Ax_2 \in D \) for \( x_1, x_2 \in C \). For any \( \lambda \in [0, 1] \), then \( \lambda Ax_1 + (1 - \lambda) Ax_2 = A(\lambda x_1 + (1 - \lambda) x_2) \). Since \( \lambda x_1 + (1 - \lambda) x_2 \in C \) because \( C \) is convex, by definition \( A(\lambda x_1 + (1 - \lambda) x_2) \in D \). Thus, \( D \) is convex.

Consider \( D' = \{ x : Ax \in C' \} \). Consider \( x_1, x_2 \in D' \). Then \( A(\lambda x_1 + (1 - \lambda) x_2) = \lambda Ax_1 + (1 - \lambda) Ax_2 \in C' \) since \( Ax_1, Ax_2 \in C' \) and \( C' \) is convex. Thus, \( D' \) is convex.

3. Let \( C \subseteq \mathbb{R}^n \) be a closed convex set. Define \( C^* = \{ y \in \mathbb{R}^n : y^T x \leq 1 \ \forall x \in C \} \). This set is called the polar of \( C \). Show that:

(i) \( C^* \) is a convex set containing the origin.

Solution: First show \( C^* \) is convex. Let \( y_1, y_2 \in C^* \); thus,

\[
y_j \cdot x \leq 1 \text{ for all } x \in C \text{ and } j = 1, 2
\]
For any \( \lambda \in [0, 1] \), consider \( \bar{y} = \lambda y_1 + (1-\lambda)y_2 \). For any \( x \in C \), \( \bar{y} \cdot x = (\lambda y_1 + (1-\lambda)y_2) \cdot x = \lambda (y_1 \cdot x) + (1-\lambda)(y_2 \cdot x) \leq \lambda + (1-\lambda) = 1 \), where the inequality comes from (1).

Next, since \( 0 \cdot x = 0 \leq 1 \) for all \( x \in C, 0 \in C^* \).

(ii) If \( 0 \in C \), then \((C^*)^* = C\).

Solution: By definition, \((C^*)^* = \{ z \in \mathbb{R}^n : z \cdot y \leq 1, \forall y \in C^* \} \). Clearly, \( C \subseteq (C^*)^* \) since for any \( x \in C \), \( x \cdot y = y \cdot x \leq 1 \) for all \( y \in C^* \). Consider \( x^* \notin C \). By the separating hyperplane theorem, there exists \( c \in \mathbb{R}^n \) and \( \delta \in \mathbb{R} \) such that \( c \cdot x \leq \delta \) for all \( x \in C \) and \( c \cdot x^* > \delta \). Since \( 0 \in C \), \( 0 = c \cdot 0 \leq \delta \). Consider two cases now.

Case 1: \( \delta > 0 \). Then \( \frac{1}{\delta} c \cdot x \leq 1 \) for all \( x \in C \) and so \( \frac{1}{\delta} c \in C^* \), but \( x^* \cdot \frac{1}{\delta} c = \frac{1}{\delta} c \cdot x^* > 1 \) and so \( x^* \notin (C^*)^* \).

Case 2: \( \delta = 0 \). Let \( \mu = \frac{1}{2} (c \cdot x^*) > 0 \). Then \( \frac{1}{\mu} c \cdot x \leq \frac{1}{\mu} \delta = 0 \leq 1 \) for all \( x \in C \). Thus, \( \frac{1}{\mu} c \in C^* \). However, \( \frac{1}{\mu} c \cdot x^* = 2 > 1 \). Thus, \( x^* \cdot \frac{1}{\mu} c > 1 \) and so \( x^* \notin (C^*)^* \).

Thus, in both cases, \( x^* \notin (C^*)^* \) and so \((C^*)^* \subseteq C \).

4. (Problems 2.15, 2.16, 2.17) Prove the following Farkas’ type results.

(i) Prove that there exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if for each \( y \geq 0 \), \( y^T A \geq 0 \Rightarrow y^T b \geq 0 \).

Solution: There exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if there is a solution to \( A' \begin{bmatrix} x \\ s \end{bmatrix} = b \) and \( x, s \geq 0 \), where \( A' = [A \quad I] \) where \( I \) is the \( m \times m \) identity matrix.

By Farkas’ Lemma, this system has a solution if and only if there does not exist \( y \in \mathbb{R}^m \) such that \( y^T [A \quad I] \geq 0 \) and \( y^T b < 0 \). In other words, \( y^T [A \quad I] \geq 0 \Rightarrow y^T b \geq 0 \). Finally, observe that \( y^T [A \quad I] \) is simply \( y \geq 0, y^T A \geq 0 \).

(ii) Prove that there exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if for each \( y \in \mathbb{R}^m, y^T A \geq 0 \Rightarrow y^T A = 0 \).

Solution: There exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if there exists a solution to \( Ax = 0, x \geq 1 \) where \( 1 \) is the all one’s vector in \( \mathbb{R}^n \). This is because any \( x > 0 \) with \( Ax = 0 \) can be scaled to get \( \bar{x} = \frac{1}{\mu} x \) where \( \mu \) is the smallest component of \( x \).

\( Ax = 0, x \geq 1 \) has a solution if and only if \( A' \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( x, s \geq 0 \) has a solution where \( A' = \begin{bmatrix} A & 0 \\ I & -I \end{bmatrix} \). By Farkas’ lemma, this has a solution if and only if there does not exist \( y \in \mathbb{R}^m, z \in \mathbb{R}^n \) such that \( [y \quad z]^T A' \geq 0 \) and \( z^T 1 < 0 \). This means there exists no \( y, z \) such that \( y^T A + z \geq 0 \) and \( z \leq 0 \) and \( \sum_{i=1}^n z_i < 0 \). In other words, any \( y, z \) satisfying \( y^T A + z \geq 0 \) and \( z \leq 0 \) must have \( z = 0 \). This is equivalent to saying that for any \( y \in \mathbb{R}^m \), \( y^T A \geq 0 \) implies \( y^T A = 0 \).

(iii) Prove that there exists \( x \neq 0 \) satisfying \( x \geq 0 \) and \( Ax = 0 \) if and only if there is no vector \( y \in \mathbb{R}^m \) satisfying \( y^T A > 0 \).

Solution: There exists \( x \neq 0 \) satisfying \( x \geq 0 \) and \( Ax = 0 \) if and only if \( Ax = 0, 1^T x = 1, x \geq 0 \) has a solution (by scaling \( x \) by \( \frac{1}{\sum_{i=1}^n z_i} \)). Thus, by Farkas’ lemma, this happens if and only if there is no \( y \in \mathbb{R}^m, \mu \in \mathbb{R} \) such that \( y^T A + \mu 1^T \geq 0 \) and \( \mu < 0 \). This is equivalent to the existence of \( y \in \mathbb{R}^m \) such that \( y^T A > 0 \).