AMS 553.766: Combinatorial Optimization
Homework Problems - Week IV

For any two vectors $x, y \in \mathbb{R}^d$, $x \leq y$ means the inequality holds for each component: $x_i \leq y_i$ for all $i = 1, \ldots, d$. Similarly, $x < y$ means $x_i < y_i$ for all $i = 1, \ldots, d$.

For the following problems, $A \in \mathbb{R}^{m \times n}$ is an $m \times n$ matrix, and $b \in \mathbb{R}^m$.

1. Let $X$ be an arbitrary (possibly infinite) subset of $\mathbb{R}^n$. The convex hull of $X$, denoted by $\text{conv}(X)$, is a convex set $C$ such that $X \subseteq C$ and for any other convex set $C'$, $X \subseteq C' \Rightarrow C \subseteq C'$, i.e., the convex hull is the smallest (with respect to set inclusion) convex set containing $X$. Show that

$$\text{conv}(X) = \bigcap \{ C : X \subseteq C, C \text{ convex} \} = \{ \lambda_1 x_1 + \ldots + \lambda_t x_t : x_1, \ldots, x_t \in X, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1 \}$$

Solution: Let $\hat{C} = \bigcap \{ C : X \subseteq C, C \text{ convex} \}$. Consider any other convex set $C'$ such that $X \subseteq C'$. Then $C'$ appears in the intersection, and thus $\hat{C} \subseteq C'$. Thus, $\hat{C} = \text{conv}(X)$.

Next, let $\tilde{C} = \{ \lambda_1 x_1 + \ldots + \lambda_t x_t : x_1, \ldots, x_t \in X, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1 \}$. Then,

(a) $\tilde{C}$ is convex. Consider two points $z_1, z_2 \in \tilde{C}$. Thus there exist two finite index sets $I_1, I_2$; two finite subsets of $X$ given by $X_1 = \{ x_i \in X : i \in I_1 \}$ and $X_2 = \{ x_i \in X : i \in I_2 \}$; and two subsets of nonnegative real numbers $\{ \lambda_i^1 \geq 0, i \in I_1 \}$; $\{ \lambda_i^2 \geq 0, i \in I_2 \}$ such that $\sum_{i \in I_j} \lambda_i^j = 1$ for $j = 1, 2$, with the following property: $z_j = \sum_{i \in I_j} \lambda_i^j x_i^j$ for $j = 1, 2$. Then for any $\lambda \in [0, 1]$, $\lambda z_1 + (1-\lambda) z_2 = \lambda (\sum_{i \in I_1} \lambda_i^1 x_i^1) + (1-\lambda) (\sum_{i \in I_2} \lambda_i^2 x_i^2)$. Consider the finite set $\tilde{X} = X_1 \cup X_2$, and for each $x \in \tilde{X}$, if $x = x_i \in X_1$ with $i \in I_1$ let $\mu_x = \lambda \cdot \lambda_i^1$, and if $x = x_i \in X_2$ with $i \in I_2$, let $\mu_x = (1-\lambda) \cdot \lambda_i^2$. It is easy to check that $\sum_{x \in \tilde{X}} \mu_x = 1$, and $\lambda z_1 + (1-\lambda) z_2 = \sum_{x \in \tilde{X}} \mu_x x$. Thus, $\lambda z_1 + (1-\lambda) z_2 \in \tilde{C}$.

(b) $X \subseteq \hat{C}$. We simply use $\lambda = 1$ as the multiplier for a point from $X$.

(c) Let $C'$ be any convex set such that $X \subseteq C'$. Since $C'$ is convex, every point of the form $\lambda_1 x_1 + \ldots + \lambda_t x_t$ where $x_1, \ldots, x_t \in X, \lambda_i \geq 0, \sum_{i=1}^t \lambda_i = 1$ belongs to $C'$. Thus, $\hat{C} \subseteq C'$.

From (a), (b) and (c), we get that $\hat{C} = \text{conv}(X)$.

2. (Problem 2.2 in Schrijver’s notes) Let $C$ be a convex set in $\mathbb{R}^n$, and let $A$ be any $m \times n$ matrix. Show that the set $\{ Ax : x \in C \}$ is convex. [Thus, convexity is preserved under linear transformations.] Let $C'$ be a convex set in $\mathbb{R}^m$. Show that $\{ x \in \mathbb{R}^n : Ax \in C' \}$ is also a convex set.

Solution: Let $D = \{ Ax : x \in C \}$. Consider $Ax_1, Ax_2 \in D$ for $x_1, x_2 \in C$. For any $\lambda \in [0, 1]$, then $\lambda Ax_1 + (1-\lambda) Ax_2 = A(\lambda x_1 + (1-\lambda) x_2)$. Since $\lambda x_1 + (1-\lambda) x_2 \in C$ because $C$ is convex, by definition $\lambda Ax_1 + (1-\lambda) Ax_2 \in D$. Thus, $D$ is convex.

Consider $D' = \{ x : Ax \in C' \}$. Consider $x_1, x_2 \in D'$. Then $A(\lambda x_1 + (1-\lambda) x_2) = \lambda Ax_1 + (1-\lambda) Ax_2 \in C'$ since $Ax_1, Ax_2 \in C'$ and $C'$ is convex. Thus, $D'$ is convex.

3. Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Define $C^* = \{ y \in \mathbb{R}^n : y^T \cdot x \leq 1 \ \forall x \in C \}$. This set is called the polar of $C$. Show that:

(i) $C^*$ is a convex set containing the origin.

Solution: First show $C^*$ is convex. Let $y_1, y_2 \in C^*$; thus,

$$y_j \cdot x \leq 1 \text{ for all } x \in C \text{ and } j = 1, 2$$ (1)
For any \( \lambda \in [0,1] \), consider \( \bar{y} = \lambda y_1 + (1-\lambda)y_2 \). For any \( x \in C \), \( \bar{y} \cdot x = (\lambda y_1 + (1-\lambda)y_2) \cdot x = \lambda (y_1 \cdot x) + (1-\lambda)(y_2 \cdot x) \leq \lambda + (1-\lambda) = 1 \), where the inequality comes from (1).

Next, since \( 0 \cdot x = 0 \leq 1 \) for all \( x \in C \), \( 0 \in C^* \).

(ii) If \( 0 \in C \), then \( (C^*)^* = C \).

Solution: By definition, \( (C^*)^* = \{ z \in \mathbb{R}^n : z \cdot y \leq 1, \forall y \in C^* \} \). Clearly, \( C \subseteq (C^*)^* \) since for any \( x \in C \), \( x \cdot y = y \cdot x \leq 1 \) for all \( y \in C^* \). Consider \( x^* \not\in C \). By the separating hyperplane theorem, there exists \( c \in \mathbb{R}^n \) and \( \delta \in \mathbb{R} \) such that \( c \cdot x \leq \delta \) for all \( x \in C \) and \( c \cdot x^* > \delta \). Since \( 0 \in C \), \( 0 = c \cdot 0 \leq \delta \). Consider two cases now.

Case 1: \( \delta > 0 \). Then \( \frac{1}{\delta} c \cdot x \leq 1 \) for all \( x \in C \) and so \( \frac{1}{\delta} c \in C^* \), but \( x^* \cdot \frac{1}{\delta} c = \frac{1}{\delta} c \cdot x^* > 1 \) and so \( x^* \not\in (C^*)^* \).

Case 2: \( \delta = 0 \). Let \( \mu = \frac{1}{2}(c \cdot x^*) > 0 \). Then \( \frac{1}{\mu} c \cdot x \leq \frac{1}{\mu} \delta = 0 \leq 1 \) for all \( x \in C \). Thus, \( \frac{1}{\mu} c \in C^* \). However, \( \frac{1}{\mu} c \cdot x^* = 2 > 1 \). Thus, \( x^* \cdot \frac{1}{\mu} c > 1 \) and so \( x^* \not\in (C^*)^* \).

Thus, in both cases, \( x^* \not\in (C^*)^* \) and so \( (C^*)^* \subseteq C \).

4. (Problems 2.15, 2.16, 2.17) Prove the following Farkas’ type results.

(i) Prove that there exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if for each \( y \geq 0 \), \( y^T A \geq 0 \) \( \Rightarrow y^T b \geq 0 \).

Solution: There exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if there is a solution to \( A' \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b \) and \( x, s \geq 0 \), where \( A' = [A \ I] \) where \( I \) is the \( m \times m \) identity matrix.

By Farkas’ Lemma, this system has a solution if and only if there does not exist \( y \in \mathbb{R}^m \) such that \( y^T [A \ I] \geq 0 \) and \( y^T b < 0 \). In other words, \( y^T [A \ I] \geq 0 \) \( \Rightarrow y^T b \geq 0 \). Finally, observe that \( y^T [A \ I] \) is simply \( y^T \geq 0 \), \( y^T A \geq 0 \).

(ii) Prove that there exists \( x > 0 \) satisfying \( Ax = 0 \) if and only if for each \( y \in \mathbb{R}^m \), \( y^T A \geq 0 \) \( \Rightarrow y^T A = 0 \).

Solution: There exists \( x > 0 \) satisfying \( Ax = 0 \) if and only if there exists a solution to \( Ax = 0, x \geq 1 \) where \( 1 \) is the all one’s vector in \( \mathbb{R}^n \). This is because any \( x > 0 \) with \( Ax = 0 \) can be scaled to get \( x = \frac{1}{\mu} x \) where \( \mu \) is the smallest component of \( x \).

\( Ax = 0, x \geq 1 \) has a solution if and only if \( A' \cdot \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( x, s \geq 0 \) has a solution where \( A' = \begin{bmatrix} A & 0 \\ I & -I \end{bmatrix} \). By Farkas’ lemma, this has a solution if and only if there does not exist \( y \in \mathbb{R}^m, z \in \mathbb{R}^n \) such that \( [y \ z]^T A' \geq 0 \) and \( z^T 1 < 0 \). This means there exists no \( y, z \) such that \( y^T A + z \geq 0 \) and \( z \leq 0 \) and \( \sum_{i=1}^n z_i < 0 \). In other words, any \( y, z \) satisfying \( y^T A + z \geq 0 \) and \( z \leq 0 \) must have \( z = 0 \). This is equivalent to saying that for any \( y \in \mathbb{R}^m \), \( y^T A \geq 0 \) implies \( y^T A = 0 \).

(iii) Prove that there exists \( x \neq 0 \) satisfying \( x \geq 0 \) and \( Ax = 0 \) if and only if there is no vector \( y \in \mathbb{R}^m \) satisfying \( y^T A > 0 \).

Solution: There exists \( x \neq 0 \) satisfying \( x \geq 0 \) and \( Ax = 0 \) if and only if \( Ax = 0, 1^T x = 1, x \geq 0 \) has a solution (by scaling \( x \) by \( \frac{1}{\sum_{i=1}^n z_i} \)). Thus, by Farkas’ lemma, this happens if and only if there is no \( y \in \mathbb{R}^m, \mu \in \mathbb{R} \) such that \( y^T A + \mu 1^T \geq 0 \) and \( \mu < 0 \). This is equivalent to the existence of \( y \in \mathbb{R}^m \) such that \( y^T A > 0 \).