AMS 553.766: Combinatorial Optimization  
Homework Problems - Week IV

For any two vectors $x, y \in \mathbb{R}^d$, $x \leq y$ means the inequality holds for each component: $x_i \leq y_i$ for all $i = 1, \ldots, d$. Similarly, $x < y$ means $x_i < y_i$ for all $i = 1, \ldots, d$.

For the following problems, $A \in \mathbb{R}^{m \times n}$ is an $m \times n$ matrix, and $b \in \mathbb{R}^m$.

1. Let $X$ be an arbitrary (possibly infinite) subset of $\mathbb{R}^n$. The convex hull of $X$, denoted by $\text{conv}(X)$, is a convex set $C$ such that $X \subseteq C$ and for any other convex set $C'$, $X \subseteq C' \Rightarrow C \subseteq C'$, i.e., the convex hull is the smallest (with respect to set inclusion) convex set containing $X$. Show that

$$\text{conv}(X) = \bigcap \{C : X \subseteq C, C \text{ convex} \} = \{\lambda_1 x_1 + \ldots + \lambda_t x_t : x_1, \ldots, x_t \in X, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1\}$$

**Solution:** Let $\hat{C} = \bigcap \{C : X \subseteq C, C \text{ convex} \}$. Consider any other convex set $C'$ such that $X \subseteq C'$. Then $C'$ appears in the intersection, and thus $\hat{C} \subseteq C'$. Thus, $\hat{C} = \text{conv}(X)$.

Next, let $\hat{C} = \{\lambda_1 x_1 + \ldots + \lambda_t x_t : x_1, \ldots, x_t \in X, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1\}$. Then,

(a) $\hat{C}$ is convex. Consider two points $z_1, z_2 \in \hat{C}$. Thus there exist two finite index sets $I_1, I_2$, two finite subsets of $X$ given by $X_1 = \{x_i^1 \in X : i \in I_1\}$ and $X_2 = \{x_i^2 \in X : i \in I_2\}$, and two subsets of nonnegative real numbers $\{\lambda_i^1 \geq 0, i \in I_1\}, \{\lambda_i^2 \geq 0, i \in I_2\}$ such that $\sum_{i \in I_j} \lambda_i^j = 1$ for $j = 1, 2$, with the following property: $z_j = \sum_{i \in I_j} \lambda_i^j x_i^j$ for $j = 1, 2$. Then for any $\lambda \in [0, 1], \lambda z_1 + (1-\lambda) z_2 = \lambda (\sum_{i \in I_1} \lambda_i^1 x_i^1) + (1-\lambda) (\sum_{i \in I_2} \lambda_i^2 x_i^2)$. Consider the finite set $\hat{X} = X_1 \cup X_2$, and for each $x \in \hat{X}$, if $x = x_i \in X_1$ with $i \in I_1$ let $\mu_x = \lambda \cdot \lambda_i^1$, and if $x = x_i \in X_2$ with $i \in I_2$, let $\mu_x = (1-\lambda) \cdot \lambda_i^2$. It is easy to check that $\sum_{x \in \hat{X}} \mu_x x = 1$, and $\lambda z_1 + (1-\lambda) z_2 = \sum_{x \in \hat{X}} \mu_x x$. Thus, $\lambda z_1 + (1-\lambda) z_2 \in \hat{C}$.

(b) $X \subseteq \hat{C}$. We simply use $\lambda = 1$ as the multiplier for a point from $X$.

(c) Let $C'$ be any convex set such that $X \subseteq C'$. Since $C'$ is convex, every point of the form $\lambda_1 x_1 + \ldots + \lambda_t x_t$ where $x_1, \ldots, x_t \in X, \lambda_i \geq 0, \sum_{i=1}^t \lambda_i = 1$ belongs to $C'$. Thus, $\hat{C} \subseteq C'$.

From (a), (b) and (c), we get that $\hat{C} = \text{conv}(X)$.

2. (Problem 2.2 in Schrijver’s notes) Let $C$ be a convex set in $\mathbb{R}^n$, and let $A$ be any $m \times n$ matrix. Show that the set $\{Ax : x \in C\}$ is convex. [Thus, convexity is preserved under linear transformations.] Let $C'$ be a convex set in $\mathbb{R}^m$. Show that $\{x \in \mathbb{R}^n : Ax \in C'\}$ is also a convex set.

**Solution:** Let $D = \{Ax : x \in C\}$. Consider $Ax_1, Ax_2 \in D$ for $x_1, x_2 \in C$. For any $\lambda \in [0, 1]$, then $\lambda Ax_1 + (1-\lambda) Ax_2 = A(\lambda x_1 + (1-\lambda) x_2)$. Since $\lambda x_1 + (1-\lambda) x_2 \in C$ because $C$ is convex, by definition $\lambda Ax_1 + (1-\lambda) Ax_2 \in D$. Thus, $D$ is convex.

Consider $D' = \{x : Ax \in C'\}$. Consider $x_1, x_2 \in D'$. Then $A(\lambda x_1 + (1-\lambda) x_2) = \lambda Ax_1 + (1-\lambda) Ax_2 \in C'$ since $Ax_1, Ax_2 \in C'$ and $C'$ is convex. Thus, $D'$ is convex.

3. Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Define $C^* = \{y \in \mathbb{R}^n : y^T \cdot x \leq 1 \ \forall x \in C\}$. This set is called the polar of $C$. Show that:

(i) $C^*$ is a convex set containing the origin.

**Solution:** First show $C^*$ is convex. Let $y_1, y_2 \in C^*$; thus,

$$y_j \cdot x \leq 1 \text{ for all } x \in C \text{ and } j = 1, 2 \quad (1)$$
For any \( \lambda \in [0, 1] \), consider \( \bar{y} = \lambda y_1 + (1 - \lambda) y_2 \). For any \( x \in C \), \( \bar{y} \cdot x = (\lambda y_1 + (1 - \lambda) y_2) \cdot x = \lambda (y_1 \cdot x) + (1 - \lambda) (y_2 \cdot x) \leq \lambda + (1 - \lambda) = 1 \), where the inequality comes from (1).

Next, since \( 0 \cdot x = 0 \leq 1 \) for all \( x \in C \), \( 0 \in C^* \).

(ii) If \( 0 \in C \), then \((C^*)^* = C\).

Solution: By definition, \((C^*)^* = \{ z \in \mathbb{R}^n : z \cdot y \leq 1, \forall y \in C^* \} \). Clearly, \( C \subseteq (C^*)^* \) since for any \( x \in C \), \( x \cdot y = y \cdot x \leq 1 \) for all \( y \in C^* \). Consider \( x^* \notin C \). By the separating hyperplane theorem, there exists \( c \in \mathbb{R}^n \) and \( \delta \in \mathbb{R} \) such that \( c \cdot x \leq \delta \) for all \( x \in C \) and \( c \cdot x^* > \delta \). Since \( 0 \in C \), \( 0 = c \cdot 0 \leq \delta \). Consider two cases now.

Case 1: \( \delta > 0 \). Then \( \frac{1}{\delta} c \cdot x \leq 1 \) for all \( x \in C \) and so \( \frac{1}{\delta} c \in C^* \), but \( x^* \cdot \frac{1}{\delta} c = \frac{1}{\delta} c \cdot x^* > 1 \) and so \( x^* \notin (C^*)^* \).

Case 2: \( \delta = 0 \). Let \( \mu = \frac{1}{2} (c \cdot x^*) > 0 \). Then \( \frac{1}{\mu} c \cdot x \leq \frac{1}{\mu} \delta = 0 \) for all \( x \in C \). Thus, \( \frac{1}{\mu} c \in C^* \). However, \( \frac{1}{\mu} c \cdot x^* = 2 > 1 \). Thus, \( x^* \cdot \frac{1}{\mu} c > 1 \) and so \( x^* \notin (C^*)^* \).

Thus, in both cases, \( x^* \notin (C^*)^* \) and so \((C^*)^* \subseteq C\).