AMS 550.666: Combinatorial Optimization
Homework Problems - Week V

For any two vectors \( x, y \in \mathbb{R}^d \), \( x \leq y \) means the inequality holds for each component: \( x_i \leq y_i \) for all \( i = 1, \ldots, d \). Similarly, \( x < y \) means \( x_i < y_i \) for all \( i = 1, \ldots, d \).

For the following problems, \( A \in \mathbb{R}^{m \times n} \) is an \( m \times n \) matrix, and \( b \in \mathbb{R}^m \).

1. Let \( X \) be an arbitrary (possibly infinite) subset of \( \mathbb{R}^n \). The convex hull of \( X \), denoted by \( \text{conv}(X) \), is a convex set \( C \) such that \( X \subseteq C \) and for any other convex set \( C' \), \( X \subseteq C' \Rightarrow C \subseteq C' \), i.e., the convex hull is the smallest (with respect to set inclusion) convex set containing \( X \). Show that

\[
\text{conv}(X) = \bigcap \{(C: X \subseteq C, C \text{ convex}) = \{\lambda_1 x_1 + \ldots + \lambda_t x_t : x_1, \ldots, x_t \in X, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1\} \]

Solution: Let \( \hat{C} = \bigcap \{(C : X \subseteq C, C \text{ convex}) \). Consider any other convex set \( C' \) such that \( X \subseteq C' \). Then \( C' \) appears in the intersection, and thus \( \hat{C} \subseteq C' \). Thus, \( \hat{C} = \text{conv}(X) \).

Next, let \( \tilde{C} = \{\lambda_1 x_1 + \ldots + \lambda_t x_t : x_1, \ldots, x_t \in X, \lambda_1, \ldots, \lambda_t \geq 0, \sum_{i=1}^t \lambda_i = 1\} \). Then,

(a) \( \tilde{C} \) is convex. Consider two points \( z_1, z_2 \in \tilde{C} \). Thus there exist two finite index sets \( I_1, I_2 \), two finite subsets of \( X \) given by \( X_1 = \{x_i^1 \in X : i \in I_1\} \) and \( X_2 = \{x_i^2 \in X : i \in I_2\} \), and two subsets of nonnegative real numbers \( \{\lambda_i^1 \geq 0, i \in I_1\}, \{\lambda_i^2 \geq 0, i \in I_2\} \) such that \( \sum_{i \in I_1} \lambda_i^1 = 1 \) for \( j = 1, 2 \), with the following property: \( z_j = \sum_{i \in I_j} \lambda_i^j x_i^j \) for \( j = 1, 2 \). Then for any \( \lambda \in [0, 1] \), \( \lambda z_1 + (1-\lambda) z_2 = \lambda (\sum_{i \in I_1} \lambda_i^1 x_i^1) + (1-\lambda) (\sum_{i \in I_2} \lambda_i^2 x_i^2) \). Consider the finite set \( \tilde{X} = X_1 \cup X_2 \), and for each \( x \in \tilde{X} \), if \( x = x_i \in X_1 \) with \( i \in I_1 \) let \( \mu_x = \lambda \cdot \lambda_i^1 \), and if \( x = x_i \in X_2 \) with \( i \in I_2 \), let \( \mu_x = (1-\lambda) \cdot \lambda_i^2 \). It is easy to check that \( \sum_{x \in \tilde{X}} \mu_x = 1 \), and \( \lambda z_1 + (1-\lambda) z_2 = \sum_{x \in \tilde{X}} \mu_x x \). Thus, \( \tilde{C} = \{x \in \tilde{X} : x \in \tilde{C}\} \).

(b) \( X \subseteq \hat{C} \). We simply use \( \lambda = 1 \) as the multiplier for a point from \( X \).

(c) Let \( C' \) be any convex set such that \( X \subseteq C' \). Since \( C' \) is convex, every point of the form \( \lambda_1 x_1 + \ldots + \lambda_t x_t \) where \( x_1, \ldots, x_t \in X, \lambda_i \geq 0, \sum_{i=1}^t \lambda_i = 1 \) belongs to \( C' \). Thus, \( \hat{C} \subseteq C' \).

From (a), (b) and (c), we get that \( \hat{C} = \text{conv}(X) \).

2. (Problem 2.2 in Schrijver’s notes) Let \( C \) be a convex set in \( \mathbb{R}^n \), and let \( A \) be any \( m \times n \) matrix. Show that the set \( \{Ax : x \in C\} \) is convex. [Thus, convexity is preserved under linear transformations.] Let \( C' \) be a convex set in \( \mathbb{R}^m \). Show that \( \{x \in \mathbb{R}^n : Ax \in C'\} \) is also a convex set.

Solution: Let \( D = \{Ax : x \in C\} \). Consider \( Ax_1, Ax_2 \in D \) for \( x_1, x_2 \in C \). For any \( \lambda \in [0, 1] \), then \( \lambda Ax_1 + (1-\lambda) Ax_2 = A(\lambda x_1 + (1-\lambda) x_2) \). Since \( Ax_1 + (1-\lambda) x_2 \in C \) because \( C \) is convex, by definition \( \lambda Ax_1 + (1-\lambda) Ax_2 \in D \). Thus, \( D \) is convex.

Consider \( D' = \{x : Ax \in C'\} \). Consider \( x_1, x_2 \in D' \). Then \( A(\lambda x_1 + (1-\lambda) x_2) = \lambda Ax_1 + (1-\lambda) Ax_2 \in C' \) since \( Ax_1, Ax_2 \in C' \) and \( C' \) is convex. Thus, \( D' \) is convex.

3. Let \( C \) be a (topologically) closed, convex set. Show that there exists a (potentially infinite) family of halfspaces given by \( c_i^j \in \mathbb{R}^n, \delta_i \in \mathbb{R} \) indexed by \( i \in I \), such that

\[
C = \{x \in \mathbb{R}^n : c_i^j \cdot x \leq \delta_i \ \forall i \in I\}.
\]
4. Let $C \subseteq \mathbb{R}^n$ be a closed convex set. Define $C^* = \{y \in \mathbb{R}^n : y^T \cdot x \leq 1 \ \forall x \in C\}$. This set is called the polar of $C$. Show that:

(i) $C^*$ is a convex set containing the origin.

Solution: First show $C^*$ is convex. Let $y_1, y_2 \in C^*$; thus,

$$y_j \cdot x \leq 1 \text{ for all } x \in C \text{ and } j = 1, 2 \tag{1}$$

For any $\lambda \in [0, 1]$, consider $\bar{y} = \lambda y_1 + (1-\lambda)y_2$. For any $x \in C$, $\bar{y} \cdot x = (\lambda y_1 + (1-\lambda)y_2) \cdot x = \lambda(y_1 \cdot x) + (1-\lambda)(y_2 \cdot x) \leq \lambda + (1-\lambda) = 1$, where the inequality comes from (1).

Next, since $0 \cdot x = 0 \leq 1$ for all $x \in C$, $0 \in C^*$.

(ii) If $0 \in C$, then $(C^*)^* = C$.

Solution: By definition, $(C^*)^* = \{z \in \mathbb{R}^n : z \cdot y \leq 1, \forall y \in C^*\}$. Clearly, $C \subseteq (C^*)^*$ since for any $x \in C$, $x \cdot y = y \cdot x \leq 1$ for all $y \in C^*$. Consider $x^* \notin C$. By the separating hyperplane theorem, there exists $c \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$ such that $c \cdot x \leq \delta$ for all $x \in C$ and $c \cdot x^* > \delta$. Since $0 \in C$, $0 = c \cdot 0 \leq \delta$. Consider two cases now.

Case 1: $\delta > 0$. Then $\frac{1}{\delta} c \cdot x \leq 1$ for all $x \in C$ and so $\frac{1}{\delta} c \in C^*$, but $x^* \cdot \frac{1}{\delta} c = \frac{1}{\delta} c \cdot x^* > 1$ and so $x^* \notin (C^*)^*$.

Case 2: $\delta = 0$. Let $\mu = \frac{1}{\mu} c \cdot x^* > 0$. Then $\frac{1}{\mu} c \cdot x \leq \frac{1}{\mu} \delta = 0 \leq 1$ for all $x \in C$. Thus, $\frac{1}{\mu} c \in C^*$. However, $\frac{1}{\mu} c \cdot x^* = 2 > 1$. Thus, $x^* \cdot \frac{1}{\mu} c > 1$ and so $x^* \notin (C^*)^*$.

Thus, in both cases, $x^* \notin (C^*)^*$ and so $(C^*)^* \subseteq C$.

5. Suppose there exists $x \in \mathbb{R}^n$ such that $Ax < b$. Show that $P$ is full-dimensional, i.e., dimension of $P$ equals $n$.

Solution: Let $e^1, \ldots, e^n$ be the standard unit vectors in $\mathbb{R}^n$. We will show that there exists $\epsilon > 0$ such that $\{x, x + \epsilon e^1, x + \epsilon e^2, \ldots, x + \epsilon e^n\} \subseteq P$. Since these are $n+1$ affinely independent points, this will show $P$ has dimension $n$.

For every $j \in \{1, \ldots, n\}$, define

$$\epsilon_j = \min_{i \in \{1, \ldots, m\}} \{ \frac{b_i - a_i \cdot x}{a_i \cdot e_j} : a_i \cdot e_j > 0 \}$$

Observe that each $\epsilon_j > 0$ since $Ax < b$. Let $\epsilon = \min \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Consider any $j \in \{1, \ldots, n\}$ and any $i \in \{1, \ldots, m\}$. We will show that $a_i \cdot (x + \epsilon e^j) \leq b_i$. This will complete the proof. If $a_i \cdot e^j \leq 0$, then this is true since $a_i \cdot x \leq b_i$. When $a_i \cdot e^j > 0$,

$$a_i \cdot (x + \epsilon e^j) = a_i \cdot x + \epsilon (a_i \cdot e^j) \leq a_i \cdot x + \frac{b_i - a_i \cdot x}{a_i \cdot e^j} (a_i \cdot e^j) = b_i$$
6. Let \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \). Suppose that for each \( j = 1, \ldots, m \), there exists \( x^j \in P \) such that \( a^j \cdot x^j < b_j \), i.e., for every inequality, there is a point in \( P \) that satisfies this inequality strictly. Show that \( P \) is full-dimensional.

Solution: Consider the point \( x^* = \sum_{j=1}^{m} \frac{1}{m} x^j \). For any \( j^* = 1, \ldots, m \), \( a^{j^*} \cdot x^* = \sum_{j=1}^{m} \frac{1}{m} (a^{j^*} \cdot x^j) \). Since \( x^j \in P \), we have \( a^{j^*} \cdot x^j < b_j \) for \( j \neq j^* \) and \( a^{j^*} \cdot x^{j^*} < b_{j^*} \). Therefore, \( \sum_{j=1}^{m} \frac{1}{m} (a^{j^*} \cdot x^j) < b_{j^*} \). Therefore, \( Ax^* < b \) and by the previous exercise, \( P \) is full dimensional.

7. (Problems 2.15, 2.16, 2.17) Prove the following Farkas’ type results.

(i) Prove that there exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if for each \( y \geq 0 \), \( y^T A \geq 0 \Rightarrow y^T b \geq 0 \).

Solution: There exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if there is a solution to \( A' \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b \) and \( x, s \geq 0 \), where \( A' = [A \ I] \) where \( I \) is the \( m \times m \) identity matrix.

By Farkas’ Lemma, this system has a solution if and only if there does not exist \( y \in \mathbb{R}^m \) such that \( y^T [A \ I] \geq 0 \) and \( y^T b < 0 \). In other words, \( y^T [A \ I] \geq 0 \Rightarrow y^T b \geq 0 \). Finally, observe that \( y^T [A \ I] \) is simply \( y_1 \geq 0 \).

(ii) Prove that there exists \( x > 0 \) satisfying \( Ax = 0 \) if and only if for each \( y \in \mathbb{R}^m \), \( y^T A \geq 0 \Rightarrow y^T A = 0 \).

Solution: There exists \( x \geq 0 \) satisfying \( Ax \leq b \) if and only if there exists a solution to \( Ax = 0, x \geq 1 \) where \( 1 \) is the all one’s vector in \( \mathbb{R}^n \). This is because any \( x > 0 \) with \( Ax = 0 \) can be scaled to get \( \tilde{x} = \frac{1}{\mu}x \) where \( \mu \) is the smallest component of \( x \).

\( Ax = 0, x \geq 1 \) has a solution if and only if \( A' = \begin{bmatrix} A & 0 \\ I & -I \end{bmatrix} \). By Farkas’ lemma, this has a solution if and only if there does not exist \( y \in \mathbb{R}^m, z \in \mathbb{R}^n \) such that \( [y \ z]^T A' \geq 0 \) and \( z^T 1 < 0 \). This means there exists no \( y, z \) such that \( y^T A + z \geq 0 \) and \( z \leq 0 \) and \( \sum z_i < 0 \). In other words, any \( y, z \) satisfying \( y^T A + z \geq 0 \) and \( z \leq 0 \) must have \( z = 0 \). This is equivalent to saying that for any \( y \in \mathbb{R}^m, y^T A \geq 0 \) implies \( y^T A = 0 \).

(iii) Prove that there exists \( x \neq 0 \) satisfying \( x \geq 0 \) and \( Ax = 0 \) if and only if there is no vector \( y \in \mathbb{R}^m \) satisfying \( y^T A > 0 \).

Solution: There exists \( x \neq 0 \) satisfying \( x \geq 0 \) and \( Ax = 0 \) if and only if \( Ax = 0, 1^T x = 1, x \geq 0 \) has a solution (by scaling \( x \) by \( \frac{1}{\sum x_i} \)). Thus, by Farkas’ lemma, this happens if and only if there is no \( y \in \mathbb{R}^m, \mu \in \mathbb{R} \) such that \( y^T A + \mu 1^T \geq 0 \) and \( \mu < 0 \). This is equivalent to the existence of \( y \in \mathbb{R}^m \) such that \( y^T A > 0 \).

8. Let \( P = \{ x \in \mathbb{R}^n : Ax \leq b \} \) be a nonempty polyhedron. Let \( C = \{ r \in \mathbb{R}^n : x + \lambda r \in P \text{ for all } x \in P, \lambda \in \mathbb{R}_+ \} \) where \( \mathbb{R}_+ \) is the set of nonnegative real numbers. Show that

(i) \( C \) is a convex cone. [This cone is called the recession cone of the polyhedron \( P \)]

(ii) \( C = \{ r \in \mathbb{R}^n : Ar \leq 0 \} \).

Solution:

(i) We need to first verify that \( C \) is convex. Consider any \( r^1, r^2 \in C \) and any \( \mu \in [0, 1] \) and let \( \bar{r} = \mu r^1 + (1 - \mu) r^2 \). For any \( x \in P, \lambda \geq 0 \) \( x + \lambda \bar{r} = x + \lambda \mu r_1 + \lambda (1 - \mu) r_2 \). Since \( r_1 \in C, x + \lambda r_1 \in P \) and since \( r_2 \in C, (x + \lambda r_1) + \lambda (1 - \mu) r_2 \in P \). A very similar argument shows that \( C \) is a cone.

(ii) Consider any \( r \) such that \( Ar \leq 0 \). Then \( A(x + \lambda r) = Ax + \lambda Ar \leq Ax \) for \( \lambda \geq 0 \) and \( Ax \leq b \) if \( x \in P \). Thus, \( A(x + \lambda r) \leq b \) for all \( x \in P \) and \( \lambda \geq 0 \). This shows that
Consider any \( r \) such that \( Ar \not\leq b \). This means, for some \( i = 1, \ldots, m \), \( a^i \cdot r > 0 \). Since \( P \) is nonempty, let \( x \in P \). Since \( a^i \cdot r > 0 \), there exists \( \lambda \geq 0 \) such that \( a^i(x + \lambda r) = a^i \cdot x + \lambda(a^i \cdot r) > b_i \). This shows that \( r \not\in C \). Hence \( C \subseteq \{ r \in \mathbb{R}^n : Ar \leq 0 \} \).

Let \( L = \{ r \in \mathbb{R}^n : x + \lambda r \in P \text{ for all } x \in P, \lambda \in \mathbb{R} \} \). Show that

(i) \( L \) is a linear subspace of \( \mathbb{R}^n \). [This is called the lineality space of the polyhedron \( P \)]

(ii) \( L = \{ r \in \mathbb{R}^n : Ar = 0 \} \).

Solution:

(i) For any \( r_1, r_2 \in L \), \( x + \lambda(r_1 + r_2) = x + \lambda r_1 + \lambda r_2 \). Since \( r_1 \in L \), \( x + \lambda r_1 \in P \) and since \( r_2 \in L \), \( (x + \lambda r_1) + \lambda r_2 \in P \). Thus \( r_1 + r_2 \in L \). Similarly, for any \( r \in L \) and \( \mu \in \mathbb{R} \), \( x + \lambda(\mu r) = x + (\lambda \mu)r \in P \) for any \( \lambda \in \mathbb{R} \) since \( r \in L \). Thus, \( \mu r \in L \). Therefore, \( L \) is a linear subspace.

(ii) Proof is similar to the case of the cone.