In the following $G = (V, E)$ is an undirected graph. $\nu(G)$ will denote the size of the maximum matching in $G$. For a subset $A \subseteq V$, $\text{oc}(V \setminus A)$ denotes the number of odd connected components in $G \setminus A$.

1. (Problem 5.2 from textbook) Let $M$ be a matching in a general graph $G$ and let $p$ be the cardinality of a maximum matching. Show that there are at least $p - |M|$ vertex-disjoint $M$-augmenting paths.

2. (Problem 5.4 from textbook) Let $p > 0$ be the cardinality of the maximum matching in $G$, and let $M$ be a matching of cardinality at most $p - \sqrt{p}$. Show that there exists an $M$-augmenting path having at most $\sqrt{p}$ edges from $M$.

3. (Problem 5.5 from textbook) Let $G = (V, E)$ be a general graph and $k \leq |V|/2$ be a given positive integer. Construct a graph $G'$ such that $G'$ has a perfect matching if and only if $G$ has a matching of size $k$.

4. Show that the following two statements are equivalent:
   1. For any graph $G$, $\nu(G) = \min_{A \subseteq V} \frac{1}{2}(|V| - \text{oc}(V \setminus A) + |A|)$.
   2. For any graph $G$, $G$ has a perfect matching if and only if for every subset $A \subseteq V$, $\text{oc}(V \setminus A) \leq |A|$.

   [Hint: Use the previous exercise]

We define a vertex $v$ in $G$ to be inessential if there exists a maximum matching $M$ such that $v$ is $M$-exposed. We say that $v$ is essential if every maximum matching cover $v$.

5. Recall the Tutte-Berge formula from class: $\nu(G) = \min_{A \subseteq V} \frac{1}{2}(|V| - \text{oc}(V \setminus A) + |A|)$. Let $A^*$ be a minimizer of the right hand side in the Tutte-Berge formula. Show that all vertices in $A^*$ are essential.

6. Let $M$ be a matching (not necessarily maximum) in $G$, and $T$ be an $M$-alternating tree. Suppose there is an edge $vw$ such that $v, w \in B(T)$, thus we have an odd cycle $C$ using the edge $vw$. Let $G' = G \times C$, $M' = M \setminus E(C)$ and $T' = T \times C$. Show the following: i) $M'$ is a matching in $G'$, ii) $T'$ is an $M'$-alternating tree in $G'$, and iii) $C \in B(T')$.

7. (Problem 5.15 from textbook) Let $T_1', \ldots, T_k'$ be the trees at the termination of the blossom algorithm on $G$. For any super/pseudo vertex $v$, let $S(v)$ denote all the original vertices that were shrunk into $v$ (note that the super/pseudo vertex $v$ may correspond to multiple shrinking operations - we are considering ALL the original vertices that went into $v$ in the process of these shrinking operations). Let $B = \bigcup (B(T_i') : i = 1, \ldots, k)$ and let $B' = \bigcup (S(v) : v \in B)$. Prove that $B'$ is exactly the set of inessential vertices of $G$. 

1