AMS 553.766: Combinatorial Optimization
Homework Problems - Week III

In the following $G = (V,E)$ is an undirected graph. $\nu(G)$ will denote the size of the maximum matching in $G$. For a subset $A \subseteq V$, $oc(V \setminus A)$ denotes the number of odd connected components in $G \setminus A$.

1. **Assignment Problem.** Suppose we have a set $\{J_1, J_2, \ldots, J_r\}$ of $r$ jobs to be filled by a pool of $s$ applicants $\{A_1, A_2, \ldots, A_s\}$. Each job can be filled by at most one applicant and each applicant be assigned to at most one job. Also each job can be filled by only a subset of applicants qualified for the jobs. It is known in advance if a job $J_i$ can be filled by applicant $A_j$. The goal is to find the maximum number of jobs that can be filled. Formulate this as a maximum matching problem.

Solution: Create a bipartite graph with $r$ vertices on one side, corresponding to the jobs, and $s$ vertices on the other side, corresponding to applicants. For each “job” vertex, put an edge from it to all applicant vertices that are qualified to do that job. Now, any matching corresponds to a valid assignment of applicants to jobs, and maximum matching corresponds to the maximum number of jobs getting filled.

2. (Problem 5.2 from textbook) Let $M$ be a matching in a general graph $G$ and let $p$ be the cardinality of a maximum matching. Show that there are at least $p - |M|$ node-disjoint $M$-augmenting paths.

Solution: Consider the graph $G' = (V, M \Delta N)$ where $N$ is a maximum matching. Observe that each path in $G'$ can have at most 1 more edge from $N$ compared to $M$. Since, $N$ has at least $p - |M|$ more edges, there should be at least these many paths in $G'$ with more edges from $N$. As argued in class, each of these paths is an $M$-augmenting path.

3. (Problem 5.4 from textbook) Let $p > 0$ be the cardinality of the maximum matching in $G$, and let $M$ be a matching of cardinality at most $p - \sqrt{p}$. Show that there exists an $M$-augmenting path having at most $\sqrt{p}$ edges from $M$.

Solution: Using the previous problem, we have at least $p - |M| \geq p - (p - \sqrt{p}) = \sqrt{p}$ vertex-disjoint $M$-augmenting paths. If all of these paths have strictly greater than $\sqrt{p}$ edges from $M$, then $M$ has strictly more than $p$ edges, contradicting that $M$ has at most $p - \sqrt{p}$ edges.

4. A **line** in a matrix is a row or column of the matrix. Show that the minimum number of lines to cover all nonzero entries of a matrix (not necessarily square) is equal to the maximum number of nonzero entries, no two of which lie in a common line.

Solution: Consider a bipartite graph $G = (A \cup B, E(G))$ where $A$ corresponds to the rows of the matrix, and $B$ corresponds to the columns and there is an edge between two vertices if the corresponding entry in the matrix is nonzero. The minimum number of lines to cover all nonzero entries is thus a minimum vertex cover. The maximum number of non zero entries no two of which are in a common line forms a set of matching edges. The result now follows from König’s theorem for bipartite graphs.

A **perfect matching** in a graph $G$ is a matching that covers all vertices (and thus, the graph has an even number of vertices).

5. **Structure of difference of matchings.**

   (i) Let $M, N$ be two maximum matchings in $G$. Describe the structure of $G' := (V(G), M \Delta N)$.
Solution: $G'$ is a disjoint union of even sized cycles, even sized paths and some isolated vertices. We know the cycles are even sized from the result in class. The paths have to be even sized otherwise we have an augmenting path for one of the matchings, contradicting its maximumness.

(ii) Let $M, N$ be two perfect matchings in $G$. Describe the structure of $G' := (V(G), M\Delta N)$.
Solution: $G'$ is a collection of even sized cycles and isolated vertices. We cannot have a path in $G'$ because then each of the end points of the path would not be matched in one of the matchings, contradicting the fact that we have a perfect matching.

(iii) Show that a tree can have at most one perfect matching.
Solution: Suppose to the contrary that we have two distinct perfect matchings $M, N$. By part (ii), the set difference graph is a collection of cycles and isolated vertices. Since $M$ and $N$ are distinct, we cannot be left with only isolated vertices. Therefore, we have at least one cycle contradicting the fact that it is a tree.

6. Let $M$ be a matching in a graph $G$. Show there exists a maximum matching that covers every vertex covered by $M$. Deduce that in a graph with no isolated vertices, every vertex is covered by some maximum matching.
Solution: Let $\mathcal{N}$ be the set of all matchings that cover all the vertices covered by $M$ (and maybe more vertices). Let $N$ be a matching in $\mathcal{N}$ that is of largest size. If $N$ is not a maximum matching in $G$, there exists an $N$-augmenting path $P$. Then $N\Delta P$ is a matching with one more edge and also covers all the vertices covered by $N$. In particular, it covers all the vertices covered by $M$ and therefore $N\Delta P$ is in $\mathcal{N}$. This contradicts the choice of $N$.

7. A matching $M$ is maximal if for every edge $e \in E(G) \setminus M$, $M \cup \{e\}$ is not a matching. In other words, there is no edge in the graph that can be added to $M$ and form a matching.

(i) Give examples to show that a maximal matching need not be a maximum matching.
Solution: Consider the following graph:

![Graph Image]

The edge $\{2, 4\}$ is a maximal matching but not a maximum matching which is of size 2.

(ii) Suppose $M$ is a maximal matching. Show that $|M| \geq \frac{\min - \text{vertex-cover}(G)}{2}$.
Solution: Consider a maximum matching $N$. If there exists an edge $e$ in $N$ such that both end points are left unmatched by $M$, then $M \cup \{e\}$ is a matching, contradicting the maximality of $M$. Thus, the set $S$ of endpoints of the edges in $M$ form a vertex cover. Therefore, $2 \cdot |M| = |S| \geq \text{min - vertex - cover}(G)$. The result follows by dividing through by 2.

8. Let $M$ be a matching (not necessarily maximum) in $G$, and $T$ be an $M$-alternating tree. Suppose there is an edge $vw$ such that $v, w \in B(T)$, thus we have an odd cycle $C$ using the edge $vw$. Let $G' = G \times C$, $M' = M \setminus E(C)$ and $T' = T \times C$. Show the following: i) $M'$ is a matching in $G'$, ii) $T'$ is an $M'$-alternating tree in $G'$, and iii) $C \in B(T')$. 

Solution: Let \( p \) be the vertex in \( C \) that is closest to the root of \( T \). Note that \( p \) belongs to \( B(T) \).

**Claim 1.** \( M \cap E(C) \) covers all the vertices in \( C \) except \( p \). Moreover, unless \( p \) is the root in \( T \), there is an edge in \( M \cap E(T) \) that covers \( p \) and this edge does not belong to \( E(C) \).

**Proof.** Now every vertex in the path from \( p \) to \( v \), except \( p \), is \( M \)-covered by an edge from \( E(C) \cap E(T) \). Similarly, every vertex in the path from \( p \) to \( w \), except \( p \), is also \( M \)-covered by an edge from \( E(C) \cap E(T) \). Thus, \( p \) is the only vertex in \( C \) that is not \( M \)-covered by an edge from \( C \). The claim about \( p \) follows because \( T \) is an \( M \)-alternating tree. \( \Box \)

i) By Claim 1, \( M \) has at most 1 edge incident on the vertices of \( C \) that does not belong to \( E(C) \). Hence, shrinking \( C \) cannot create a conflict on \( C \) in \( G' \) with respect to \( M' \). Therefore, \( M' \) is a matching in \( G' \).

ii) We need to verify the following properties:

a) Every node \( p' \in T' \) other than the root is covered by an edge of \( M' \cap E(T') \).

Suppose \( p' = C \). If \( C \) is the root in \( T' \), then nothing to check. Else, if \( C \) is not the root in \( T' \), then \( p \) is not the root in \( T \). By Claim 1, there is a matching edge \( e \) in \( M \cap E(T) \) that covers \( p \) such that \( e \notin E(C) \). Thus, \( e \in M' \cap E(T') \) and covers \( C = p' \).

If \( p' \neq C \), then \( p' \notin V(C) \). Since \( T \) is an \( M \)-alternating path, there is an edge \( e \in E(T) \) that covers \( p' \) in \( G \), so \( e \notin E(C) \). Thus, \( e \in M' \cap E(T') \) and covers \( p' \).

b) For every node \( p' \in T' \), the path in \( T' \) from the root of \( T' \) to \( p' \) is \( M' \)-alternating. If the path does not use \( C \) in \( T' \), then the path is unchanged from \( T \) to \( T' \). Since it was \( M \)-alternating in \( T \), it is \( M' \)-alternating in \( T' \). If the path \( P \) does use \( C \), the part of \( P \) from the root to \( C \) is alternating, since it did not change from \( T \) to \( T' \). Moreover, the part of \( P \) from \( C \) to \( p' \) also did not change from \( T \) to \( T' \) and is therefore \( M' \)-alternating. Since, \( M' \) is a valid matching from part i), \( P \) does alternate at \( C \). Hence, \( P \) is \( M' \)-alternating.

iii) Observe that the distance of \( C \) from the root in \( T' \) is the same as the distance of \( p \) from the root in \( T \). Since \( p \in B(T) \), this implies that \( C \in B(T') \).

9. Define vertex \( v \) in a graph \( G \) to be **inessential** if there exists a maximum matching that does not cover \( v \). Otherwise \( v \) is said to be **essential**, i.e., *every* maximum matching must cover \( v \).

- Suppose \( A \) is a minimizer of the RHS of the Tutte-Berge formula. Show that every vertex in \( A \) is essential.

**Solution:** We let \( \nu(G) \) denote the maximum matching size in any graph \( G \).

Let \( H_1, \ldots, H_k \) be the odd components in \( G \setminus A \). Consider a matching \( M \) that leaves a vertex \( v \in A \) exposed. Thus, at most \( |A| - 1 \) odd components can have edges from \( M \) going to \( A \). Each of remaining \( k - (|A| - 1) \) odd components must have at least one \( M \)-exposed vertex. Thus \( |M| \leq \frac{1}{2}(|V| - (k - (|A| - 1))) = \frac{1}{2}(|V| - k + |A|) - \frac{1}{2} = \nu(G) - \frac{1}{2} \); the second equality follows because \( A \) is the minimizer in the Tutte-Berge formula. Thus, \( |M| < \nu(G) \) and hence \( M \) cannot be a maximum matching. Thus, no maximum matching can leave any vertex in \( A \) exposed.

We also show that the vertices in any even component of \( G \setminus A \) are essential. For any matching \( M \), the number of \( |M| \)-exposed vertices equals \( |V| - 2|M| \). Since, \( \frac{1}{2}(|V| - k + |A|) = \nu(G) \), the number of exposed vertices in any maximum matching equals \( |V| - 2\nu(G) = k - |A| \). Since each \( H_i \) has edges only within itself or going to \( A \),
any matching has to leave \( k - |A| \) vertices exposed from these odd components. Thus, a maximum matching cannot leave any other vertex exposed. Thus in any even component \( D \), no vertex is exposed in a maximum matching.

- (Problem 5.15 from textbook) Let \( T_1, \ldots, T_k \) be the trees at the termination of Edmonds’ maximum matching algorithm on \( G \). For any super/pseudo vertex \( v \), let \( S(v) \) denote all the original vertices that were shrunk into \( v \) (note that the super/pseudo vertex \( v \) may correspond to multiple shrinking operations - we are considering ALL the original vertices that went into \( v \) in the process of these shrinking operations). Let \( B = \bigcup (B(T_i) : i = 1, \ldots, k) \) and let \( B' = \bigcup (S(v) : v \in B) \). Prove that \( B' \) is exactly the set of inessential vertices of \( G \).

Solution:

**Observation 1.** Let \( G \) be any graph and \( C \) be an odd cycle \( C \) in \( G \). Suppose \( M' \) is a matching in \( G' = G \times C \) that leaves \( C \) exposed. Then for any vertex \( v \in C \), there exists a matching \( M \) for \( G \) such that the number of \( M' \)-exposed nodes in \( G' \) is equal to the number of \( M \)-exposed nodes in \( G \). and \( v \) is \( M \)-exposed. This can simply be done by adding \((|V(C)| - 1)/2\) matching edges from \( C \) that do not touch \( v \in C \).

**Observation 2.** Let \( G \) be any graph, \( M \) be a matching in \( G \), and \( T \) be an \( M \)-alternating tree with no shrunk nodes in \( T \). Given any vertex \( v \in B(T) \), one can change \( M \) to a matching \( M_1 \) of the same size such that \( v \) is \( M_1 \)-exposed. This can be achieved by considering the path from the root to \( v \), and changing all the matching edges to nonmatching edges and all nonmatching edges to matching edges.

Consider any vertex \( u \in B' \). This means there exists \( i \in \{1, \ldots, k\} \) such that \( u \in S(v) \) for some \( v \in B(T_i) \). Let \( M' \) be the “unexpanded” matching found during the execution of Edmonds’ algorithm, that is expanded to give the maximum matching in \( G \). Note that \( M' \) leaves the root of \( T_i \) exposed. Thus, by Observation 2, we can change \( M' \) to a matching \( M'' \) of the same size, and thus leaves the same number of exposed nodes, such that \( v \) is \( M'' \)-exposed. Now \( v \) may have been obtained by shrinking many odd cycles in intermediate stages of the algorithm. Let call these cycles \( C_1, C_2, \ldots, C_t \) (note that many of these cycles may not be odd cycles in the original graph \( G \)). Next apply Observation 1 to expand \( v \) into \( C_t \) to get a matching \( M \) that leaves the same number of exposed nodes as \( M'' \) (and thus \( M' \)), and leaves the pseudonode that contains \( u \) exposed (which may be an original node). Iteratively repeating this expansion procedure, adding matching edges from the odd cycle that leave the pseudonode containing \( u \) exposed, we can find a matching \( M \) in \( G \) that leaves the same number of exposed nodes as \( M' \), and leaves \( u \) exposed. Since the maximum matching \( M^* \) output by Edmonds’ algorithm leaves the same number of exposed nodes as \( M' \) (namely \( k \)), the size of \( M \) equals the size of \( M^* \), and thus \( M \) is also a maximum matching that leaves \( u \) exposed.

Therefore, all vertices in \( B' \) are inessential.

Next we show that all other vertices in \( G \setminus B' \) are essential. Any such vertex is either in \( A(T_i') \) for some \( i \in \{1, \ldots, k\} \), or is a vertex in the last stage of the Edmonds’ algorithm when all vertices in \( D = V \setminus (V(T_1) \cup V(T_2) \cup \ldots \cup V(T_k)) \) are matched up. (Here, \( T_i \) is the notation introduced in the class for the graph induced by the expanded \( T_i' \)). This means that in \( G \setminus (A(T_1') \cup \ldots \cup A(T_k')) \), the vertices in \( D \) are partitioned into even components, since no matching edge used in the last stage of the Edmonds’ algorithm is incident on \( A = (A(T_1') \cup \ldots \cup A(T_k')) \). We know that \( A \) is a minimizer in the Tutte-Berge formula. Hence, using the arguments from the first part, we find that all vertices in \( A \cup D \) are essential.