In the following $G = (V, E)$ is an undirected graph. $\nu(G)$ will denote the size of the maximum matching in $G$. For a subset $A \subseteq V$, $oc(V \setminus A)$ denotes the number of odd connected components in $G \setminus A$.

1. (Problem 5.2 from textbook) Let $M$ be a matching in a general graph $G$ and let $p$ be the cardinality of a maximum matching. Show that there are at least $p - |M|$ node-disjoint $M$-augmenting paths.

Solution: Consider the graph $G' = (V, M \Delta N)$ where $N$ is a maximum matching. Observe that each path in $G'$ can have at most 1 more edge from $N$ compared to $M$. Since, $N$ has at least $p - |M|$ more edges, there should be at least these many paths in $G'$ with more edges from $N$. As argued in class, each of these paths is an $M$-augmenting path.

2. (Problem 5.4 from textbook) Let $p > 0$ be the cardinality of the maximum matching in $G$, and let $M$ be a matching of cardinality at most $p - \sqrt{p}$. Show that there exists an $M$-augmenting path having at most $\sqrt{p}$ edges from $M$.

Solution: Using the previous problem, we have at least $p - |M| \geq p - (p - \sqrt{p}) = \sqrt{p}$ vertex-disjoint $M$-augmenting paths. If all of these paths have strictly greater than $\sqrt{p}$ edges from $M$, then $M$ has strictly more than $p$ edges, contradicting that that $M$ has at most $p - \sqrt{p}$ edges.

3. (Problem 5.5 from textbook) Let $G = (V, E)$ be a general graph and $k \leq |V|/2$ be a given positive integer. Construct a graph $G'$ such that $G'$ has a perfect matching (see problem 3 above) if and only if $G$ has a matching of size $k$.

Solution: $G'$ has is constructed from $G$ be adding a set $V_{\text{slack}}$ of $n - 2k$ additional vertices ($n = |V|$). Moreover, we have additional edges from every vertex in $V_{\text{slack}}$ to every vertex in $V$. Now if $G$ has a matching $M$ of size $k$, then $n - 2k$ vertices in $G$ are unmatched. In $G'$ create a perfect matching by using $M$ and additionally matching the $M$-exposed vertices from $V$ to $V_{\text{slack}}$. On the other hand, if $G'$ has a perfect matching, then all vertices in $V_{\text{slack}}$ are matched, thus $2k$ vertices in $V$ are matched using the original edges, thus giving a matching of size $k$ in $G$.

4. Show that the following two statements are equivalent:

   1. For any graph $G$, $\nu(G) = \min_{A \subseteq V} \frac{1}{2}(|V| - oc(V \setminus A) + |A|)$.
   2. For any graph $G$, $G$ has a perfect matching if and only if for every subset $A \subseteq V$, $oc(V \setminus A) \leq |A|$.

   [Hint: Use the previous exercise]

Solution: (1. $\Rightarrow$ 2.) If $G$ has a perfect matching, then $\nu(G) = |V|/2$ and thus, $|V|/2 = \min_{A \subseteq V} \frac{1}{2}(|V| - oc(V \setminus A) + |A|) \leq \frac{1}{2}(|V| - oc(V \setminus A) + |A|)$ for every subset $A \subseteq V$. Thus, $oc(V \setminus A) \leq A$. On the other hand, if for every subset $A \subseteq V$, $oc(V \setminus A) \leq |A|$, then

$$|V|/2 \leq \min_{A \subseteq V} \frac{1}{2}(|V| - oc(V \setminus A) + |A|) = \nu(G) \leq |V|/2.$$ 

Thus, all inequalities above must be at equality, showing that $\nu(G) = |V|/2$ and $G$ has a perfect matching.
5. Recall the Tutte-Berge formula from class: $\nu(G) = k$. It suffices to show that there exists a subset $A \subseteq V$ such that $\frac{1}{2}(|V| - oc(V \setminus A) + |A|) \leq k$. Construct a graph $G' = (V' \cup E')$ from $G$ by adding a set $V_{slack}$ of $n - 2(k+1)$ additional vertices ($n = |V|$). Thus, $V' = V \cup V_{slack}$. Moreover, we have additional edges from every vertex in $V_{slack}$ to every vertex in $V$. Recall that in HW for Week III, Problem 10, it was shown that $G$ has a matching of size $k + 1$ if and only if $G'$ has a perfect matching. Thus, since the maximum matching in $G$ is of size $k$, $G'$ does not have a perfect matching. So there exists a set $A' \subseteq V \cup V_{slack}$ such that $oc(V' \setminus A') > |A'|$. Since $G'$ has an even number ($2n - 2(k+1)$) of vertices, $A'$ cannot be the empty set. Thus, $oc(V' \setminus A') \geq 2$. Thus, $V_{slack} \subseteq A'$, otherwise, $oc(V' \setminus A') = 1$ since every vertex in $V_{slack}$ is connected to every other vertex in $V$. Let $A = A' \cap V$. Thus, $oc(V' \setminus A') = oc(V \setminus A)$. Moreover, $|A'| = |A| + |V_{slack}| = |A| + n - 2(k+1)$. Since $oc(V' \setminus A') > |A'|$,

$$
\begin{align*}
oc(V \setminus A) &> |A| + n - 2(k+1) \\
\Rightarrow \quad k + 1 &> \frac{1}{2} (|V| - oc(V \setminus A) + |A|) \\
\Rightarrow \quad k &\geq \frac{1}{2} (|V| - oc(V \setminus A) + |A|)
\end{align*}
$$

The last inequality follows since the RHS is an integer (Why?).

6. Let $M$ be a matching (not necessarily maximum) in $G$, and $T$ be an $M$-alternating tree. Suppose there is an edge $vw$ such that $v, w \in B(T)$, thus we have an odd cycle $C$ using the edge $vw$. Let $G' = G \times C$, $M' = M \setminus E(C)$ and $T' = T \setminus C$. Show the following: i) $M'$ is a matching in $G'$, ii) $T'$ is an $M'$-alternating tree in $G'$, and iii) $C \in B(T')$.

Solution: Let $p$ be the vertex in $C$ that is closest to the root of $T$. Note that $p$ belongs to $B(T)$.

**Claim 1.** $M \cap E(C)$ covers all the vertices in $C$ except $p$. Moreover, unless $p$ is the root in $T$, there is an edge in $M \cap E(T)$ that covers $p$ and this edge does not belong to $E(C)$.

**Proof.** Now every vertex in the path from $p$ to $v$, except $p$, is $M$-covered by an edge from $E(C) \cap E(T)$. Similarly, every vertex in the path from $p$ to $w$, except $p$, is also $M$-covered by an edge from $E(C) \cap E(T)$. Thus, $p$ is the only vertex in $C$ that is not $M$-covered by an edge from $C$. The claim about $p$ follows because $T$ is an $M$-alternating tree.

i) By Claim 1, $M$ has at most 1 edge incident on the vertices of $C$ that does not belong to $E(C)$. Hence, shrinking $C$ cannot create a conflict on $C$ in $G'$ with respect to $M'$. Therefore, $M'$ is a matching in $G'$.

ii) We need to verify the following properties :

a) Every node $p' \in T'$ other than the root is covered by an edge of $M' \cap E(T')$. 

Suppose $p' = C$. If $C$ is the root in $T'$, then nothing to check. Else, if $C$ is not the root in $T'$, $p$ is not the root in $T$. By Claim 1, there is a matching edge $e$ in $M \cap E(T)$ that covers $p$ such that $e \notin E(C)$. Thus, $e \in M' \cap E(T')$ and covers $C = p'$.

If $p' \neq C$, then $p' \notin V(C)$. Since $T$ is an $M$-alternating path, there is an edge $e \in E(T)$ that covers $p'$ in $G$, so $e \notin E(C)$. Thus, $e \in M' \cap E(T')$ and covers $p'$.

b) For every node $p' \in T'$, the path in $T'$ from the root of $T'$ to $p'$ is $M'$-alternating. If the path does not use $C$ in $T'$, then the path is unchanged from $T$ to $T'$. Since it was $M$-alternating in $T$, it is $M'$-alternating in $T'$. If the path $P$ does use $C$, the part of $P$ from the root to $C$ is alternating, since it did not change from $T$ to $T'$. Moreover, the part of $P$ from $C$ to $p'$ also did not change from $T$ to $T'$ and is therefore $M'$-alternating. Since, $M'$ is a valid matching from part i), $P$ does alternate at $C$. Hence, $P$ is $M'$-alternating.

c) Observe that the distance of $C$ from the root in $T'$ is the same as the distance of $p$ from the root in $T$. Since $p \in B(T)$, this implies that $C \in B(T')$.

7. (Problem 5.15 from textbook) Let $T'_1, \ldots, T'_k$ be the trees at the termination of the blossom algorithm on $G$. For any shrunk/pseudo node $v$, let $S(v)$ denote all the original nodes that were shrunk into $v$ (note that the shrunk/pseudo node $v$ may correspond to multiple shrinking operations - we are considering ALL the original nodes that went into $v$ in the process of these shrinking operations). Let $B = \bigcup (B(T'_i) : i = 1, \ldots, k)$ and $B' = \bigcup (S(v) : v \in B)$. Prove that $B'$ is exactly the set of inessential nodes of $G$.

Solution:

**Observation 1.** Let $G$ be any graph and $C$ be an odd cycle $C$ in $G$. Suppose $M'$ is a matching in $G' = G \times C$ that leaves $C$ exposed. Then for any vertex $v \in C$, there exists a matching $M$ for $G$ such that the number of $M'$-exposed nodes in $G'$ is equal to the number of $M$-exposed nodes in $G$, and $v$ is $M$-exposed. This can simply be done by adding $(|V(C)| - 1)/2$ matching edges from $C$ that do not touch $v \in C$.

**Observation 2.** Let $G$ be any graph, $M$ be a matching in $G$, and $T$ be an $M$-alternating tree (there are no shrunk nodes in $T$). Given any vertex $v \in B(T)$, one can change $M$ to a matching $M_1$ of the same size such that $v$ is $M_1$-exposed. This can be achieved by considering the path from the root to $v$, and changing all the matching edges to nonmatching edges and all nonmatching edges to matching edges.

Consider any vertex $u \in B'$. This means there exists $i \in \{1, \ldots, k\}$ such that $u \in S(v)$ for some $v \in B(T'_i)$. Let $M'$ be the “unexpanded” matching found during the execution of the blossom algorithm, that is expanded to give the maximum matching in $G$. Note that $M'$ leaves the root of $T'_i$ exposed. Thus, by Observation 2, we can change $M'$ to a matching $M''$ of the same size, and thus leaves the same number of exposed nodes, such that $v$ is $M''$-exposed. Now $v$ may have been obtained by shrinking many odd cycles in intermediate stages of the algorithm. Let’s call these cycles $C_1, C_2, \ldots, C_t$ (note that many of these cycles may not be odd cycles in the original graph $G$). Next apply Observation 1 to expand $v$ into $C_t$ to get a matching $M$ that leaves the same number of exposed nodes as $M''$ (and thus $M'$), and leaves the pseudonode that contains $u$ exposed (which may be an original node). Iteratively repeating this expansion procedure, adding matching edges from the odd cycle that leave the pseudonode containing $u$ exposed, we can find a matching $M$ in $G$ that leaves the same number of exposed nodes as $M'$, and leaves $u$ exposed. Since the maximum matching $M^*$ output by the Blossom algorithm leaves the same number of exposed nodes as $M'$ (namely $k$), the size of $M$ equals the size of $M^*$, and thus $M$ is also a maximum matching that leaves $u$ exposed.
Therefore, all vertices in $B'$ are inessential.

Next we show that all other vertices in $G \setminus B'$ are essential. Any such vertex is either in $A(T'_i)$ for some $i \in \{1, \ldots, k\}$, or is a vertex in the last stage of the Blossom algorithm when all vertices in $D = V \setminus (V(T_1) \cup V(T_2) \cup \ldots \cup V(T_k))$ are matched up. (Here, $T_i$ is the notation introduced in the class for the graph induced by the expanded $T'_i$). This means that in $G \setminus (A(T'_1) \cup \ldots \cup A(T'_k))$, the vertices in $D$ are partitioned into even components, since no matching edge used in the last stage of the Blossom algorithm is incident on $A = (A(T'_1) \cup \ldots \cup A(T'_k))$. We know that $A$ is a minimizer in the Tutte-Berge formula. Hence, using the result of Problem 3, we find that all vertices in $A \cup D$ are essential.