AMS 550.666: Combinatorial Optimization
Homework Problems - Week X

1. Subadditive function Cuts
   Recall the family of functions $\phi_f : \mathbb{R} \to \mathbb{R}$ defined in class parameterized by $f \in (0, 1)$:
   \[
   \phi_f(r) = \begin{cases} 
   \frac{r}{f} & [r] \leq f \\
   \frac{1-r}{1-f} & [r] > f 
   \end{cases}
   \]

   (i) Show that each $\phi_f$ is subadditive.
   (ii) Show that $\phi_f(-r) = \phi_{1-f}(r)$. (This means the family is closed under reflection around the y axis)
   (iii) [Sriram’s construction] Consider the function $\psi : \mathbb{R} \to \mathbb{R}$ defined as
   \[
   \psi(r) = \begin{cases} 
   \sin(\pi [r]) & 0 \leq [r] < \frac{1}{2} \\
   2 - \sin(\pi (1-[r])) & \frac{1}{2} \leq [r] < 1 
   \end{cases}
   \]
   Show that $\psi$ is periodic with period 1 and subadditive.

2. Semidefinite Programs
   (i) Let $S_n \subseteq \mathbb{R}^{n^2}$ be the set of all $n \times n$ positive semidefinite matrices. Show that $S_n$ is a convex cone in $\mathbb{R}^{n^2}$, i.e., for any PSD matrix $A$, $\lambda A$ is also PSD for all $\lambda \geq 0$, and for any PSD matrices $A, B$, $A + B$ is also PSD.
   (ii) Show that any linear program of the form $\max \{c^T x : Ax = b, x \geq 0\}$ can be solved by writing an equivalent semidefinite program.

3. Rank constraints are not convex. Show that the set
   \[
   \{X \in \mathbb{R}^{n^2} : X \text{ is PSD, rank}(X) \leq k\}
   \]
   is not a convex set for any $1 \leq k < n$.

4. Relaxation of Stable Set. Given a graph $G = (V, E)$, with $V = \{1, \ldots, n\}$, show that
   \[
   STAB(G) := \text{conv}\left( \left\{ x \in \mathbb{R}^n : \begin{array}{c}
   x_i + x_j \leq 1 \\
   x_i \in \{0,1\} \\
   \forall ij \in E, \forall i = 1, \ldots, n
   \end{array} \right\} \right) \subseteq \left\{ \text{Proj}_x(X) : \begin{array}{c}
   X_{00} = 1, \\
   X_{ii} = X_{i0} \quad \forall i = 1, \ldots, n, \\
   X_{ij} = 0 \quad \forall ij \in E \\
   X \text{ is PSD}
   \end{array} \right\},
   \]
   where $\text{Proj}_x(X) := (X_{01}, X_{02}, \ldots, X_{0n})$. 