AMS 553.766: Combinatorial Optimization
Homework Problems - Week X

1. Recall that a subgroup $S \subseteq \mathbb{R}^n$ is a lattice of $\mathbb{R}^n$, provided that there exists $\epsilon > 0$ such that for any point $y \in S$, $B(y, \epsilon) \cap S = \{y\}$, i.e. the ball centered at $y$ with radius $\epsilon$ contains only $y$ from $S$. Prove that a subgroup $S$ is a lattice if there exists $\epsilon > 0$ such that the ball $B(0, \epsilon)$ centered at 0 with radius $\epsilon$ does not contain any non zero elements from $S$, i.e. $B(0, \epsilon) \cap S = \{0\}$. [In other words, it is enough to check the discreteness condition at the origin only]

2. Let $a^1, \ldots, a^k$ be a set of vectors in $\mathbb{R}^n$ with only rational entries. We do not assume that the vectors are linearly independent or that $k \leq n$. Is it true that the set

$$\Lambda = \{\mu_1 a^1 + \ldots + \mu_k a^k : \mu_1, \ldots, \mu_k \in \mathbb{Z}\}$$

is a lattice of $\mathbb{R}^n$? [Hint: Think about the denominators of the entries of any vector in the above set.]

3. Hermite Normal Form
All matrices considered in this problem will be assumed to have integer entries. Consider the following elementary column operations for a matrix $A$.

i. exchanging two columns.

ii. multiplying a column by -1.

iii. adding an integral multiple of one column to another column.

a) Let $A$ be any $d \times k$ matrix and let $A'$ is obtained from $A$ by performing a finite number of sequential elementary operations as above. Show that $A$ can be obtained from $A'$ by performing elementary column operations and $Z(A) = Z(A')$, where $Z(A)$ for any matrix $A$ is defined as the set of $\{\mu_1 a^1 + \ldots + \mu_k a^k : \mu_1, \ldots, \mu_k \in \mathbb{Z}\}$, where $a^1, \ldots, a^k$ are the columns of $A$.

b) Show that any $d \times k$ integral matrix $A$ with full row rank (so $k \geq d$) can be converted into the form $[B, 0]$ using only the above elementary column operations, such that all of the following hold:

– $B$ is a nonsingular, lower triangular matrix with non-zero entries on the main diagonal.

– $B$ has only nonnegative entries.

– In each row of $B$ there is a unique maximum element, which sits on the main diagonal of $B$.

$[B, 0]$ is called the Hermite Normal Form (HNF) of $A$.

c) Let $U$ be a unimodular matrix (recall this means that $U$ is an integral $n \times n$ matrix and has determinant $\pm 1$). Show that the HNF of $U$ is the identity matrix. Conclude that the lattice generated by the columns of a unimodular matrix is $\mathbb{Z}^n$. 

1