A *dipath* in a directed graph is a *directed* path, meaning that the edges used are all “forward” edges; more precisely, a dipath is an alternating sequence of vertices and edges $v_1e_1v_2e_2 \ldots v_ke_kv_{k+1}$ such that each $e_i$ is an edge from $v_i$ to $v_{i+1}$.

1. Find the maximum flow and the minimum cut in the following graph:

![Graph 1](image1.png)

2. (Problem 3.2 from text) Find the maximum flow and the minimum cut in the following graph:

![Graph 2](image2.png)

3. Give an example of a directed graph with nonnegative *integer* capacities where there is a maximum flow with non-integer flow values (We showed in class that there is always an integer optimum solution. Consequently, your example must have multiple optimum solutions).

4. (Problem 3.3 from textbook) Suppose we allow for infinite capacities in a maximum flow problem, i.e., $c_e = \infty$ for some edge $e$ indicates that one can put any amount of flow $x_e$ on it. Prove that the maximum flow is infinite if and only if there is an $r-s$ dipath (i.e., a path with all forward edges starting at $r$ and ending at $s$) where all edges have capacity $c_e = \infty$.

5. (Problem 3.11 from textbook) Given an $x$-augmenting path $P$, the $x$-width of $P$ is defined to be $\min\{\min\{c_e - x_e : e \text{ is forward in } P\}, \min\{x_e : e \text{ is backward in } P\}\}$. Suppose that for a feasible flow $x$, there is no $x$-augmenting path of $x$-width greater than $K$. Prove that $f_x(s)$ is within $Km$ of the max flow value, where $m$ equals the total number of edges in the graph. [Hint : How would you show this for $K = 0$?]
6. (Problem 3.7 from textbook) Suppose \( \delta(R_1) \) and \( \delta(R_2) \) are both minimum cuts. Show that \( \delta(R_1 \cap R_2) \) and \( \delta(R_1 \cup R_2) \) are also minimum cuts.

7. (Problem 3.10 from textbook) Suppose that we perform an augmentation on an augmenting path of maximum \( x \)-width. Can the maximum \( x \)-width with respect to the new flow be larger?

8. Let \( G = (V, A) \) be a directed graph. The edge-connectivity \( \gamma(G, r, s) \) is the least number of edges that need to be removed from \( G \) so that there is no dipath from \( r \) to \( s \). Let \( \theta(G, r, s) \) denote the maximum number of dipaths from \( r \) to \( s \) so that no two dipaths share an edge. Clearly, \( \theta(G, r, s) \leq \gamma(G, r, s) \). Prove Menger’s theorem for edge connectivity: \( \theta(G, r, s) = \gamma(G, r, s) \). [Hint: Use the max flow-min cut theorem]

In any (directed or undirected) graph, \( G = (V, E) \), and any subset \( S \subseteq V \), \( G \setminus S \) will denote the graph formed by deleting the vertices in \( S \), and all the edges incident on them.

9. (Problem 3.42 from textbook) Two dipaths in a digraph are said to be \textit{internally vertex disjoint} if the only vertices they have in common are their initial or final vertices. A set \( S \subseteq V \) \textit{separates} \( r \) from \( s \) if there is no \((r, s)\)-dipath in \( G \setminus S \). Prove Menger’s theorem for node connectivity: For any pair of distinct vertices \( r, s \in V \), the maximum number of internally vertex disjoint \((r, s)\)-dipaths in \( G \) is equal to the minimum size of a set that separates \( r \) from \( s \). [Hint: Create a new digraph and use the max flow-min cut theorem]