1. Find the maximum flow and the minimum cut in the following graph:

![Graph 1](image1.png)

Flow values are in green, Cut vertices are circled in red.

2. (Problem 3.2 from text) Find the maximum flow and the minimum cut in the following graph:

The problem posted was slightly different from Problem 3.2 from the textbook. Here is the solution for the problem posted:

![Graph 2](image2.png)

Flow values are in green, Cut vertices are circled in red.

The solution for Problem 3.2 from the textbook is at the top of the next page.

3. (Problem 3.3 from textbook) Prove that the maximum flow is infinite if and only if there is an \((r, s)-\)dipath (i.e., a path with all forward edges) where all edges have capacity \(u_e = \infty\).

Solution: Clearly, if there is such a dipath, the max flow is infinite. We show that if there is no such dipath, then the maximum flow value is \(< \infty\). Define the following set of vertices:

\[ R = \{ v \in V : \exists \text{ dipath from } r \text{ to } v \text{ where all edges have } \infty \text{ capacity} \}. \]
By assumption, $s \notin R$ and therefore $R$ is an $(r,s)$-cut. Moreover, every edge $e \in \delta(R)$ has capacity $u_e < \infty$, otherwise, the head of that edge $e$ should have been in $R$. We know that any flow value $f_e(s) \leq u(\delta(R)) < \infty$, and thus the max flow value is finite.

4. (Problem 3.7 from textbook) Suppose $\delta(R_1)$ and $\delta(R_2)$ are both minimum cuts. Show that $\delta(R_1 \cap R_2)$ and $\delta(R_1 \cup R_2)$ are also minimum cuts.

Solution: We first claim the following.

**Claim 1.** $\delta(R_1 \cup R_2) + \delta(R_1 \cap R_2) \leq \delta(R_1) + \delta(R_2)$

**Proof.** We simply account for every edge in $E$ and show that either its capacity contributes to both sides of the inequality, or only to the right hand side. Consider any edge $e = vw$ in the digraph.

**Case 1:** $v$ is in $V \setminus (R_1 \cup R_2)$. Then $e$ does not contribute to any of the four cuts, so it contributes zero on both sides of the inequality.

**Case 2:** $v \in R_1 \setminus R_2$. Then we consider three subcases. If $w \in R_1$, then $e$ does not contribute to any of the four cuts, so it contributes zero on both sides of the inequality. If $w \notin R_1 \cup R_2$, then $e$ contributes to $\delta(R_1)$ and $\delta(R_1 \cup R_2)$, and contributes zero to $\delta(R_2)$ and $\delta(R_1 \cap R_2)$. If $w \in R_2 \setminus R_1$, then again $e$ contributes only to $\delta(R_1)$, so it contributes only to the right hand side of the inequality.

**Case 3:** $v \in R_1 \cap R_2$. We consider four subcases. If $w \notin R_1 \cup R_2$, $e$ contributes to all four cuts, and thus contributes equally to both sides of the inequality. If $w \in R_1 \cap R_2$ then $e$ does not contribute to any of the four cuts, so it contributes zero on both sides of the inequality. If $w \in R_1 \setminus R_2$, then $e$ contributes to $\delta(R_2)$ and $\delta(R_1 \cap R_2)$ and zero to the other cuts, thus contributing equally to both sides of the inequality. Finally, if $w \in R_2 \setminus R_1$, then $e$ contributes to $\delta(R_1)$ and $\delta(R_1 \cap R_2)$ and zero to the other cuts, thus contributing equally to both sides of the inequality.

**Case 4:** Similar to Case 2.

Note that the slack in the inequality only comes from Cases 2 and 4. □

Since $\delta(R_1) = \delta(R_2) = c$ where $c$ is the min cut value, and $\delta(R_1 \cap R_2) \geq c$ and $\delta(R_1 \cup R_2) \geq c$, we have $2c = c + c \leq \delta(R_1 \cup R_2) + \delta(R_1 \cap R_2) \leq \delta(R_1) + \delta(R_2) = c + c = 2c$. Therefore, $\delta(R_1 \cup R_2) + \delta(R_1 \cap R_2) = 2c$. Since $\delta(R_1 \cap R_2) \geq c$ and $\delta(R_1 \cup R_2) \geq c$, this implies $\delta(R_1 \cap R_2) = c$ and $\delta(R_1 \cup R_2) = c$. Thus, they are also min cuts.
5. (Problem 3.11 from textbook) Given an $x$-augmenting path $P$, the $x$-width of $P$ is defined to be $\min\{\min\{u_e - x_e : e \text{ is forward in } P\}, \min\{x_e : e \text{ is backward in } P\}\}$. Suppose that for a feasible flow $x$, there is no $x$-augmenting path of $x$-width greater than $K$. Prove that $f_x(s)$ is within $Km$ of the max flow value. [Hint: How would you show this for $K = 0$ ?]

Solution: Define the following set of vertices:

$$R = \{v \in V : \exists \text{ dipath from } r \text{ to } v \text{ with } x\text{-width strictly more than } K\}.$$ 

By assumptions, $s \notin R$ and so $R$ is an $(r, s)$-cut. Notice that for any edge $e \in \delta(R)$, $u_e - x_e \leq K$ because otherwise, the head of $e$ should be in $R$. Similarly, for any edge $e \in \delta(V \setminus R)$, $x_e \leq K$ because otherwise the tail of $e$ should be in $R$. We know that $f_x(s) = x(\delta(R)) - x(\delta(V \setminus R))$.

Therefore,

$$f_x(s) + x(\delta(V \setminus R)) + u(\delta(R)) - x(\delta(R)) = u(\delta(R))$$

$$f_x(s) + \sum_{e \in \delta(V \setminus R)} x_e + \sum_{e \in \delta(R)}(u_e - x_e) = u(\delta(R))$$

$$f_x(s) + \sum_{e \in \delta(V \setminus R)} K + \sum_{e \in \delta(R)} K \geq u(\delta(R))$$

$$f_x(s) + Km \geq u(\delta(R))$$

$$\geq \text{ max flow value}$$

The first inequality follows from the observation above; the second inequality follows from the fact there at most $m$ summation terms because there are at most $m$ edges in the graph; the third inequality simply says that the max flow is at most the total capacity $u(\delta(R))$ of the $(r, s)$-cut $R$.

6. The edge-connectivity $\gamma(G, r, s)$ is the least number of edges that need to be removed from $G$ so that there is no dipath from $r$ to $s$. Let $\theta(G, r, s)$ denote the maximum number of dipaths from $r$ to $s$ so that no two dipaths share an edge. Clearly, $\theta(G, r, s) \leq \gamma(G, r, s)$. Prove Menger’s theorem for edge connectivity: $\theta(G, r, s) = \gamma(G, r, s)$.

Solution: We set up a max flow problem to find $\theta(G, r, s)$. The graph is the same, with $r$ as the source, $s$ as the sink. Put a capacity of 1 for all edges. Then,

**Claim 2.** $\theta(G, r, s)$ is equal to the max flow value.

**Proof.** First, we show that the max flow is $\geq \theta(G, r, s)$. Simply send put a flow of value 1 on all edges in the $\theta(G, r, s)$-paths; since these dipaths are edge-disjoint, we have a feasible flow.

Next, we show $\theta(G, r, s)$ is $\geq$ max flow. If max flow is zero, then this is trivial, so we assume nonzero max flow $k \geq 1$. Consider an integral max flow $x^r$ (we know one exists since all capacities are integers) and consider all the edges with nonzero flow on them. We show that these edges form $f_{x^r}(s)$ (max flow value) dipaths from $r$ to $s$. Since each $x^r_e$ is an integer, $x^r_e = 1$ for all edges with non zero value. We pick any edge $e = rv$ with tail $r$ such that $x^r_e = 1$ (at least one exists since max flow is assumed to be nonzero). If $v \neq s$, by flow feasibility, there must exist an edge $e' = vu'$ such that $x^r_{e'} = 1$. Continuing in this way, we must ultimately end up at $s$. This gives a dipath from $r$ to $s$. Reducing the flow by 1 on all these edges, we reduce the max flow value by 1. Thus, iterating this process, we find at least $k$ dipaths which are by construction edge-disjoint.

Also, observe that the min-cut value is exactly equal to $\gamma(G, r, s)$: For any cut $R$, $\delta(R)$ is a separating set of edges. On the other hand, consider any separating set of edges $E'$, then the set of vertices $R = \{v \in V : \exists \text{ dipath from } r \text{ to } v\}$ contains $r$ and does not contain $s$ and thus forms an $(r, s)$-cut of total capacity at most $|E'|$.

Thus, $\theta(G, r, s) = \text{ max flow value} = \min \text{ cut value} = \gamma(G, r, s)$. 

7. (Problem 3.42 from textbook) Two paths in a digraph are said to be *internally node disjoint* if the only nodes they have in common are their initial or final nodes. A set $S \subseteq V$ separates $r$ from $s$ if there is no $(r,s)$-dipath in $G \setminus S$. Prove Menger’s theorem for node connectivity: The maximum number of internally node disjoint paths in $G$ is equal to the minimum size of a set that separates $r$ from $s$. [Hint: Create a new digraph and use max flow-min cut]

Solution sketch: Consider a digraph $G’ = (V’E’)$ defined as followed. For every $v \in V$, construct two vertices $v^+, v^- \in V’$. $E’$ contains all the edges $v^+v^-$ for every $v \in V$; call this set of $n$ edges $E_{new}$. Moreover, for any edge $e = vw \in E$, we put $v^-w^+$ in $E’$.

One puts a capacity a capacity of 1 on all edges in $E_{new}$, and put a capacity of $\infty$ on all other edges in $E’$. One can then show that the max flow value equals the maximum number of node-disjoint dipaths, and the min cut value equals the minimum size of a separating set of vertices. The max flow-min cut theorem then gives the result.

8. We are given a general digraph $G$ with edge capacities $u_e \geq 0$. Also, for every node $v \in V$, we have a real number $b_v$ (not necessarily nonnegative).

i. Give an efficient algorithm to solve the following general version of the transshipment problem discussed in class. Find a set of real numbers $x_e$ for each edge $e \in E$ such that $f_x(v) = b_v$ for all $v \in V$, and $0 \leq x_e \leq u_e$. This is called a generalized flow. In class, we looked at a special case where all edges are between a vertex with $b_v \leq 0$ and a vertex with $b_v \geq 0$ (the factory-warehouse problem). Now we are considering the problem on a more general digraph.

Solution: Make a new digraph $G’ = (V \cup \{r,s\}, E \cup E_{new})$ with two additional vertices, a source $r$ and a sink $s$. Put additional edges from $r$ to $v$ of capacity $-b_v$ if $b_v < 0$, and an edge from $v$ to $s$ of capacity $b_v$ if $b_v > 0$. Call this set of edges $E_{new}$. Then, if the max flow value is equal to $\sum_{v:b_v>0} b_v$ then we have a feasible solution to the original problem. Otherwise, if the max flow is less than this value, there is no feasible solution (clearly, the max flow cannot be larger than this value). This claim can be proved by first observing that any feasible flow to the original problem gives a max flow of value $\sum_{v:b_v>0} b_v$, by putting a flow equal to capacity on all edges in $E_{new}$. Secondly, if we have a max flow of value $\sum_{v:b_v>0} b_v$, then each of the edges in $E_{new}$ have to be at full capacity, thus giving a feasible solution to the original problem, using the flow constraints.

ii. For any subset $S \subseteq V$, let $b(S) = \sum_{v \in S} b_v$. Prove that there is a feasible solution to the generalized flow problem if and only if $b(V) = 0$ and $b(S) \leq u(\delta(V \setminus S))$ for every nonempty $S \subseteq V$.

Solution: The necessity of the condition is easy to verify by simply considering the sum $\sum_{v \in S} f_x(v)$ which should equal $b(S)$.

For the sufficiency, we use the digraph constructed in part i. We show that the min cut value is $\sum_{v:b_v>0} b_v$, and so there exists a max flow of this value by the max flow-min cut theorem. Then by the argument in part i., we will be done. Consider any $(r,s)$-cut $R$ in $G’$. Let $S = V \setminus R$. Then the total capacity of the cut $R$ in $G’$ is

$$u(\delta(R)) = \sum_{e \in E_{new}} u_e + \sum_{e = vw, v \in V \cap R, w \in V \setminus R} u_e.$$

Since $b(S) \leq u(\delta(V \setminus S))$, we have that $u(\delta(R)) \geq \sum_{v \in V:b_v>0} b(v)$. Of course the cut $R = V \cup \{r\}$ has value $\sum_{v \in V:b_v>0} b(v)$ in $G’$, so this is the value of the min cut.
9. (Open Pit Mining) In the design of an open pit mine, the region under consideration is divided into 3D blocks. Then decisions are made as to which blocks will be excavated. The only constraint is that if a block needs to be excavated, then any block “above” it also needs to be excavated (with a suitable definition for “above”). This leads to the following abstraction:

Consider a digraph \( G = (V, E) \) where each node had an associated weight (benefit) \( b_v \in \mathbb{R} \) (not necessarily nonnegative). We want to find a subset of nodes to maximize our benefit. However, if choose a node \( v \) and there is an edge \( vw \in E \), then \( w \) must also be chosen. So the goal is to find a subset of vertices \( S \subseteq V \) such that \( \delta(S) = \emptyset \) and the total benefit \( b(S) \) is maximized.

Give an efficient algorithm to solve this problem.

Solution: Make a new digraph \( G' = (V \cup \{r, s\}, E \cup E_{\text{new}}) \) with two additional vertices, a source \( r \) and a sink \( s \). Put additional edges from \( r \) to \( v \) of capacity \( b_v \) if \( b_v > 0 \), and an edge from \( v \) to \( s \) of capacity \( -b_v \) if \( b_v < 0 \). Call this set of edges \( E_{\text{new}} \). Put a capacity of \( \infty \) on all other edges.

Now solve the min cut problem on this graph. Let \( R^* = S^* \cup \{r\} \) be a minimum cut in \( G' \), where \( S^* \subseteq V \). Clearly, no edge from \( E \) belongs to \( \delta(R^*) \) since they are of infinite capacity, and there is a cut of value \( \sum_{v : b_v > 0} b_v \). We claim that the optimal set of nodes is \( S^* \). To show this consider any \((r, s)\)-cut \( R = S \cup \{r\} \). Then,

\[
    u(\delta(R)) = \sum_{v \in V \setminus S, b_v > 0} (b_v) + \sum_{v \in S, b_v < 0} (-b_v)
    = \sum_{v \in V \setminus S, b_v > 0} b_v + \sum_{v \in S, b_v > 0} b_v - \sum_{v \in S, b_v > 0} b_v + \sum_{v \in S, b_v < 0} (-b_v)
    = \sum_{v : b_v > 0} (-b_v) + \sum_{v \in S} (-b_v)
\]

Since, the first term of the right hand side of the last equation is a constant, minimizing \( u(\delta(R)) \) is equivalent to maximizing \( b(S) = \sum_{v \in S} b_v \).