A dipath in a directed graph is a directed path, meaning that the edges used are all “forward” edges; more precisely, a dipath is alternating sequence of vertices and edges $v_1 e_1 v_2 e_2 \ldots v_k e_k v_{k+1}$ such that each $e_i$ is an edge from $v_i$ to $v_{i+1}$.

1. Find the maximum flow and the minimum cut in the following graph:

![Graph with flow values and cut vertices](image)

Flow values are in green, Cut vertices are circled in red.

2. (Problem 3.2 from text) Find the maximum flow and the minimum cut in the following graph:

The problem posted was slightly different from Problem 3.2 from the textbook. Here is the solution for the problem posted:

![Graph with flow values and cut vertices](image)

Flow values are in green, Cut vertices are circled in red.

The solution for Problem 3.2 from the textbook is at the top of the next page.

3. Give an example of a directed graph with nonnegative integer capacities where there is a maximum flow with non-integer flow values (We showed in class that there is always an integer optimum solution. Consequently, your example must have multiple optimum solutions).

Solution: Consider any directed graph with a directed cycle and consider any arbitrary integer capacities on the edges. For any integral maximum flow solution, we can always add 0.5 to all the edges in the directed cycle and get another flow with the same value. This would be a maximum flow with non-integer flow values on some edges.
4. (Problem 3.3 from textbook) Prove that the maximum flow is infinite if and only if there is an \((r,s)\)-dipath (i.e., a path with all forward edges) where all edges have capacity \(u_e = \infty\).

Solution: Clearly, if there is such a dipath, the max flow is infinite. We show that if there is no such dipath, then the maximum flow value is \(< \infty\). Define the following set of vertices:

\[
R = \{ v \in V : \exists \text{ dipath from } r \to v \text{ where all edges have } \infty \text{ capacity} \}.
\]

By assumption, \(s \not\in R\) and therefore \(R\) is an \((r,s)\)-cut. Moreover, every edge \(e \in \delta(R)\) has capacity \(u_e < \infty\), otherwise, the head of that edge \(e\) should have been in \(R\). We know that any flow value \(f_x(s) \leq u(\delta(R)) < \infty\), and thus the max flow value is finite.

5. (Problem 3.11 from textbook) Given an \(x\)-augmenting path \(P\), the \(x\)-width of \(P\) is defined to be \(\min\{\min\{u_e - x_e : e \text{ is forward in } P\}, \min\{x_e : e \text{ is backward in } P\}\}\). Suppose that for a feasible flow \(x\), there is no \(x\)-augmenting path of \(x\)-width greater than \(K\). Prove that \(f_x(s)\) is within \(Km\) of the max flow value. [Hint : How would you show this for \(K = 0\)?]

Solution: Define the following set of vertices:

\[
R = \{ v \in V : \exists \text{ dipath from } r \to v \text{ with x-width strictly more than } K \}.
\]

By assumptions, \(s \not\in R\) and so \(R\) is an \((r,s)\)-cut. Notice that for any edge \(e \in \delta(R)\), \(u_e - x_e \leq K\) because otherwise, the head of \(e\) should be in \(R\). Similarly, for any edge \(e \in \delta(V \setminus R)\), \(x_e \leq K\) because otherwise the tail of \(e\) should be in \(R\). We know that \(f_x(s) = x(\delta(R)) - x(\delta(V \setminus R))\).

Therefore,

\[
\begin{align*}
f_x(s) + x(\delta(V \setminus R)) + u(\delta(R)) - x(\delta(R)) &= u(\delta(R)) \\
\Rightarrow f_x(s) + \sum_{e \in \delta(V \setminus R)} x_e + \sum_{e \in \delta(R)} (u_e - x_e) &= u(\delta(R)) \\
\Rightarrow f_x(s) + \sum_{e \in \delta(V \setminus R)} K + \sum_{e \in \delta(R)} K &\geq u(\delta(R)) \\
\Rightarrow f_x(s) + Km &\geq u(\delta(R)) \\
\Rightarrow &\geq \text{ max flow value}
\end{align*}
\]

The first inequality follows from the observation above; the second inequality follows from the fact there at most \(m\) summation terms because there are at most \(m\) edges in the graph; the third inequality simply says that the max flow is at most the total capacity \(u(\delta(R))\) of the \((r,s)\)-cut \(R\).
6. (Problem 3.7 from textbook) Suppose $\delta(R_1)$ and $\delta(R_2)$ are both minimum cuts. Show that $\delta(R_1 \cap R_2)$ and $\delta(R_1 \cup R_2)$ are also minimum cuts.

Solution: We first claim the following.

**Claim 1.** $\delta(R_1 \cup R_2) + \delta(R_1 \cap R_2) \leq \delta(R_1) + \delta(R_2)$

**Proof.** We simply account for every edge in $E$ and show that either its capacity contributes to both sides of the inequality, or only to the right hand side. Consider any edge $e = vw$ in the digraph.

**Case 1:** $v$ is in $V \setminus (R_1 \cup R_2)$. Then $e$ does not contribute to any of the four cuts, so it contributes zero on both sides of the inequality.

**Case 2:** $v \in R_1 \setminus R_2$. Then we consider three subcases. If $w \in R_1$, then $e$ does not contribute to any of the four cuts, so it contributes zero on both sides of the inequality. If $w \notin R_1 \cup R_2$, then $e$ contributes to $\delta(R_1)$ and $\delta(R_1 \cup R_2)$, and contributes zero to $\delta(R_2)$ and $\delta(R_1 \cap R_2)$. If $w \in R_2 \setminus R_1$, then again $e$ contributes only to $\delta(R_1)$, so it contributes only to the right hand side of the inequality.

**Case 3:** $v \in R_1 \cap R_2$. We consider four subcases. If $w \notin R_1 \cup R_2$, $e$ contributes to all four cuts, and thus contributes equally to both sides of the inequality. If $w \in R_1 \cap R_2$ then $e$ does not contribute to any of the four cuts, so it contributes zero on both sides of the inequality. If $w \in R_1 \setminus R_2$, then $e$ contributes to $\delta(R_1)$ and $\delta(R_1 \cap R_2)$ and zero to the other cuts, thus contributing equally to both sides of the inequality. Finally, if $w \in R_2 \setminus R_1$, then $e$ contributes to $\delta(R_1)$ and $\delta(R_1 \cap R_2)$ and zero to the other cuts, thus contributing equally to both sides of the inequality.

**Case 4:** Similar to Case 2.

Note that the slack in the inequality only comes from Cases 2 and 4. \hfill \Box

Since $\delta(R_1) = \delta(R_2) = c$ where $c$ is the min cut value, and $\delta(R_1 \cap R_2) \geq c$ and $\delta(R_1 \cup R_2) \geq c$, we have $2c = c + c \leq \delta(R_1 \cup R_2) + \delta(R_1 \cap R_2) \leq \delta(R_1) + \delta(R_2) = c + c = 2c$. Therefore, $\delta(R_1 \cup R_2) + \delta(R_1 \cap R_2) = 2c$. Since $\delta(R_1 \cap R_2) \geq c$ and $\delta(R_1 \cup R_2) \geq c$, this implies $\delta(R_1 \cap R_2) = c$ and $\delta(R_1 \cup R_2) = c$. Thus, they are also min cuts.

7. Let $G = (V, A)$ be a directed graph. The edge-connectivity $\gamma(G, r, s)$ is the least number of edges that need to be removed from $G$ so that there is no dipath from $r$ to $s$. Let $\theta(G, r, s)$ denote the maximum number of dipaths from $r$ to $s$ so that no two dipaths share an edge. Clearly, $\theta(G, r, s) \leq \gamma(G, r, s)$. Prove Menger’s theorem for edge connectivity: $\theta(G, r, s) = \gamma(G, r, s)$. [Hint: Use the max flow-min cut theorem]

Solution: We set up a max flow problem to find $\theta(G, r, s)$. The graph is the same, with $r$ as the source, $s$ as the sink. Put a capacity of 1 for all edges. Then,

**Claim 2.** $\theta(G, r, s)$ is equal to the max flow value.

**Proof.** First, we show that the max flow is $\geq \theta(G, r, s)$. Simply send out a flow of value 1 on all edges in the $\theta(G, r, s)$ dipaths; since these dipaths are edge-disjoint, we have a feasible flow.

Next, we show $\theta(G, r, s)$ is $\geq$ max flow. If max flow is zero, then this is trivial, so we assume nonzero max flow $k \geq 1$. Consider an integral max flow $x^*$ (we know one exists since all capacities are integers) and consider all the edges with nonzero flow on them. We show that these edges form $f_x^*(s)$ (max flow value) dipaths from $r$ to $s$. Since each $x^*_e$ is an integer, $x^*_e = 1$ for all edges with non zero value. We pick any edge $e = rv$ with tail $r$ such that $x^*_e = 1$
(at least one exists since max flow is assumed to be nonzero). If \( v \neq s \), by flow feasibility, there must exist an edge \( e' = vv' \) such that \( x_{v'}^* = 1 \). Continuing in this way, we must ultimately end up at \( s \). This gives a dipath from \( r \) to \( s \). Redcing the flow by 1 on all these edges, we reduce the max flow value by 1. Thus, iterating this process, we find at least \( k \) dipaths which are by construction edge-disjoint.

Also, observe that the min-cut value is exactly equal to \( \gamma(G, r, s) \): For any cut \( R \), \( \delta(R) \) is a separating set of edges. On the other hand, consider any separating set of edges \( E' \), then the set of vertices \( R = \{ v \in V : \exists \text{ dipath from } r \text{ to } v \} \) contains \( r \) and does not contain \( s \) and thus forms an \((r, s)\)-cut of total capacity at most \( |E'| \).

Thus, \( \theta(G, r, s) = \text{max flow value} = \text{min cut value} = \gamma(G, r, s) \).

In any (directed or undirected) graph, \( G = (V, E) \), and any subset \( S \subseteq V \), \( G \setminus S \) will denote the graph formed by deleting the vertices in \( S \), and all the edges incident on them.

8. (Problem 3.42 from textbook) Two paths in a digraph are said to be \textit{internally node disjoint} if the only nodes they have in common are their initial or final nodes. A set \( S \subseteq V \) \textit{separates} \( r \) from \( s \) if there is no \((r, s)\)-dipath in \( G \setminus S \). Prove Menger’s theorem for node connectivity:

The maximum number of internally node disjoint paths in \( G \) is equal to the minimum size of a set that separates \( r \) from \( s \). [Hint: Create a new digraph and use max flow-min cut]

Solution sketch: Consider a digraph \( G' = (V', E') \) defined as followed. For every \( v \in V \), construct two vertices \( v^+, v^- \in V' \). \( E' \) contains all the edges \( v^+v^- \) for every \( v \in V \); call this set of \( n \) edges \( E_{\text{new}} \). Moreover, for any edge \( e = vw \in E \), we put \( v^-w^+ \) in \( E' \).

One puts a capacity a capacity of 1 on all edges in \( E_{\text{new}} \), and put a capacity of \( \infty \) on all other edges in \( E' \). One can then show that the max flow value equals the maximum number of node-disjoint dipaths, and the min cut value equals the minimum size of a separating set of vertices. The max flow-min cut theorem then gives the result.