**AMS 550.472/672: Graph Theory**
**Midterm - Spring 2016**

- **There are 5 questions on this test.**

  **Students enrolled in AMS 550.472: Hand in any 4 of these 5 questions. You will get extra credit if you hand in solutions to all 5 problems.**

  **Students enrolled in AMS 550.672: Hand in ALL 5 questions.**

- **You have to return the test to me or Elizabeth by 9:00 am on Friday, March 11, 2016. No answers will be accepted after this deadline. You can choose to send your answers back electronically before that deadline or hand them back at the beginning of class on Friday, March 11.**

  **Please hand in ONE submission - multiple submissions will not be tolerated.**

- **You are NOT allowed to discuss any problem with another human being (this includes your classmates, of course), except with Dr. Basu. This not only includes physical communication, but also electronic communication via email/google hangout/skype/chat rooms/online forums etc.**

- **You can use a computer only as a word processor or for reading documents already on your hard disk during the duration of the midterm; in particular, you CANNOT consult the internet in regards to this midterm. You CAN use any other resource like the textbook, your notes, books from the library.**

- **You CAN cite any result we have mentioned in class or from the HWs without proof. If you cite a result (e.g., from a book) that was NOT mentioned in class, you should include a complete proof of this fact.**

- **The level of rigor expected is the same as the HW solutions. Make sure you justify all your answers.**

- **There will be partial credit; e.g., if you do not prove part (i) of some question, but use part (i)'s result to prove part (ii), you can get full credit for part (ii).**

**I attest that I have completed this exam in accordance with the rules listed above.**

**Signature** ________________________________________

**Please sign and attach this sheet along with your answers.**

1. Recall that in a connected graph, the distance between two vertices $x, y$ is defined to be the length of the shortest $x - y$ path in the graph.

   Show that a connected graph is bipartite if and only if no two adjacent vertices have the same distance from any other vertex.

2. Let $G_n$ be a graph whose vertex set are all $n!$ permutations of the set $\{1, 2, \ldots, n\}$ (e.g., for $n = 3$, $G_n$ has 6 vertices: {123, 132, 213, 231, 312, 321}.) Two vertices (permutations) are

adjacent in $G_n$ if and only if they differ by interchanging a pair of entries (e.g., 123 and 132 are adjacent, but 123 and 312 are NOT adjacent in the above $n = 3$ example.) Find the following graph parameters as functions of $n$:

Number of edges of $G_n$, Minimum vertex cover size, Maximum independent set size, Maximum matching size, Minimium edge cover size.

3. Let $T$ be a tree and let $\mathcal{T}$ be a family of subtrees of $T$ (that is, subgraphs of $T$ that are also trees). Let $k \geq 2$ be any natural number. Show that either there are $k$ disjoint trees in $\mathcal{T}$ (i.e., there exist $T_1, T_2, \ldots, T_k \in \mathcal{T}$ such that $V(T_i) \cap V(T_j) = \emptyset$ for all $i \neq j$), or there exists a set $X$ of at most $k - 1$ vertices in $V(T)$ such that every tree in $\mathcal{T}$ contains at least one vertex from $X$.

4. We say a graph is $P_3$-free if a path of length two (i.e., a path on three vertices) does not appear as an *induced* subgraph. Complete and prove the following characterization of $P_3$-free graphs:

   "A graph $G$ is $P_3$-free if and only if each connected component of $G$ is a _______________."

5. Let $A_1, A_2, \ldots, A_n$ be subsets of a finite set $A$. Let $d_1, \ldots, d_n$ be nonnegative integers. Show that there exist *pairwise disjoint* subsets $D_k \subseteq A_k$ for $k = 1, \ldots, n$ with $|D_k| = d_k$ for each $k$ if and only if

$$|\bigcup_{i \in I} A_i| \geq \sum_{i \in I} d_i$$

   for every subset $I \subseteq \{1, \ldots, n\}$. [Hint: Use Hall's theorem]