AMS 550.472/672: Graph Theory
Homework Problems - Week X

Problems to be handed in on Wednesday, April 5: 3,6,11. You may use results from class or previous HWs without proof.

1. Show that the *block-cutvertex graph* of any graph is a forest.
2. Suppose G be a graph with no isolated vertices. Show that if G has no even cycles, then every block of G is an edge or an odd cycle.
3. Let $k \geq 2$. Show that in a k-connected graph any k vertices lie on a common cycle. [Hint: Induction]
4. Recall that a graph is said to be *even* if every vertex has even degree. Show that a graph is even if and only if each block is even.
5. Show that a graph is bipartite if and only if each block is bipartite.
6. Let $G_1, \ldots, G_k$ be the blocks of a simple graph G. Show that $\chi(G) = \max_{i=1,\ldots,k} \chi(G_i)$.
7. Suppose that G has no even cycles. Show that $\chi(G) \leq 3$. [Hint: Use Problem 2]
8. Suppose every edge in a graph G appears in at most one cycle. Show that $\chi(G) \leq 3$.
9. Construct a graph G that is not a complete graph, nor an odd cycle, but has a vertex ordering according to which the greedy coloring algorithm uses $\Delta(G) + 1$ colors. [Note that Brook's theorem tells us that $\chi(G) \leq \Delta(G)$.]
10. For each $k \geq 2$, construct a tree T_k with maximum degree k such that T_k has a vertex ordering with respect to which the greedy coloring gives $k + 1$ colors.
11. For every $n \geq 2$, find a bipartite graph with $2n$ vertices such that there exists a vertex ordering relative to which the greedy coloring algorithm uses n colors rather than 2 colors.
12. Show that every graph G has a vertex ordering according to which the greedy coloring algorithm uses $\chi(G)$ colors.
13. A graph with $\chi(G) = k$ is called k-chromatic. A k-chromatic graph is called *critically* k-chromatic if for every vertex v, $G \setminus \{v\}$ is $k - 1$-colorable.
    (i) Show that every k-chromatic graph has an induced subgraph that is critically k-chromatic.
    (ii) Show that every critically k-chromatic graph has minimum degree at least $k - 1$.
14. An $n \times n$ matrix with entries in $\{1, \ldots, n\}$ is called a *Latin square* if every element of $\{1, \ldots, n\}$ appears exactly once in each column and exactly once in each row. Recast the problem of constructing Latin squares as a graph coloring problem.
15. Prove that $\chi(G) = \omega(G)$ if $\overline{G}$ is bipartite ($\overline{G}$ is the complement of G – see Problem 12 in HW II). [Hint: Interpret the quantities $\chi(G)$, $\omega(G)$ in terms of graph parameters of $\overline{G}$ and use Gallai+König's theorems]
16. Give an example of an imperfect graph G (i.e., a graph that is not perfect) with $\chi(G) = \omega(G)$.
17. Suppose G is the line graph of a k-regular bipartite graph. Show that $\chi(G) = \omega(G)$. [Hint: Interpret the quantities $\chi(G)$, $\omega(G)$ in terms of graph parameters of the bipartite graph and use Gallai+König's theorems]