Problems to be handed in on Wednesday, April 5: 3,6,11. You may use results from class or previous HWs without proof.

1. Show that the block-cutvertex graph of any graph is a tree.

2. Suppose $G$ be a graph with no isolated vertices. Show that if $G$ has no even cycles, then every block of $G$ is an edge or an odd cycle.

3. Let $k \geq 2$. Show that in a $k$-connected graph any $k$ vertices lie on a common cycle. [Hint: Induction]

4. Recall that a graph is said to be even if every vertex has even degree. Show that a graph is even if and only if each block is even.

5. Show that a graph is bipartite if and only if each block is bipartite.

6. Let $G_1, \ldots, G_k$ be the blocks of a simple graph $G$. Show that $\chi(G) = \max_{i=1,\ldots,k} \chi(G_i)$.

7. Suppose that $G$ has no even cycles. Show that $\chi(G) \leq 3$. [Hint: Use Problem 2]

8. Suppose every edge in a graph $G$ appears in at most one cycle. Show that $\chi(G) \leq 3$.

9. Construct a graph $G$ that is not a complete graph, nor an odd cycle, but has a vertex ordering according to which the greedy coloring algorithm uses $\Delta(G) + 1$ colors. [Note that Brook’s theorem tells us that $\chi(G) \leq \Delta(G)$]

10. For each $k \geq 2$, construct a tree $T_k$ with maximum degree $k$ such that $T_k$ has a vertex ordering with respect to which the greedy coloring gives $k + 1$ colors.

11. For every $n \geq 2$, find a bipartite graph with $2n$ vertices such that there exists a vertex ordering relative to which the greedy coloring algorithm uses $n$ colors rather than 2 colors.

12. Show that every graph $G$ has a vertex ordering according to which the greedy coloring algorithm uses $\chi(G)$ colors.

13. A graph with $\chi(G) = k$ is called $k$-chromatic. A $k$-chromatic graph is called critically $k$-chromatic if for every vertex $v$, $G \setminus \{v\}$ is $k - 1$-colorable.

   (i) Show that every $k$-chromatic graph has an induced subgraph that is critically $k$-chromatic.

   (ii) Show that every critically $k$-chromatic graph has minimum degree at least $k - 1$.

14. An $n \times n$ matrix with entries in $\{1, \ldots, n\}$ is called a Latin square if every element of $\{1, \ldots, n\}$ appears exactly once in each column and exactly once in each row. Recast the problem of constructing Latin squares as a graph coloring problem.

15. Prove that $\chi(G) = \omega(G)$ if $G$ is bipartite ($G$ is the complement of $G$ – see Problem 12 in HW II). [Hint: Interpret the quantities $\chi(G)$, $\omega(G)$ in terms of graph parameters of $\overline{G}$ and use Gallai+König’s theorems]

16. Give an example of an imperfect graph $G$ (i.e., a graph that is not perfect) with $\chi(G) = \omega(G)$.

17. Suppose $G$ is the line graph of a $k$-regular bipartite graph. Show that $\chi(G) = \omega(G)$. [Hint: Interpret the quantities $\chi(G)$, $\omega(G)$ in terms of graph parameters of the bipartite graph and use Gallai+König’s theorems]