AMS 550.472/672: Graph Theory
Homework Problems - Week IX

Problems to be handed in on Wednesday, March 30: 4,5,6. You may use results from class or previous HWs without proof.

1. Let G be a k-vertex connected graph for some natural number $k \geq 1$. Let S and T be disjoint subsets of vertices in G, such that both S and T have at least k vertices. Show that there exist k paths $P_1, P_2, \ldots, P_k$ in G such that two things hold: 1) for each P_i, the starting vertex is in S and the end vertex is in T, 2) $V(P_i) \cap V(P_j) = \emptyset$ for $i \neq j$ (they cannot have any common vertex - not even the end points).

2. Do Problem 4.2.28 from the textbook.

3. Suppose G is a simple graph with at least 3 vertices. Show that G is 2-connected if and only if for every triple of vertices x, y, z, there is an $x - z$ path through y.

4. Consider the n-CUBE graph (see Problem 4 on HW I). Consider the following two vertices: $x = (0, 0, \ldots, 0)$ (all zeroes vector) and $y = (1, 1, \ldots, 1)$ (all ones vector). Find a set of vertex disjoint paths of maximum size between x and y.

5. **Minimally k-connected graphs.** A k-connected graph G is called *minimally k-connected*, if for every edge e, $G \setminus \{e\}$ is not k-connected. Show that

   (i) Show that if G is minimally 2-connected, then the minimum degree of G is *exactly* 2.

   (ii) Show that a minimally 2-connected graph G with at least 4 vertices has at most $2|V(G)| - 4$ edges. Construct a minimally 2 connected graph G with at least 4 vertices with exactly $2|V(G)| - 4$ edges.

   [Comment: Compare the result with minimally 1-connected graphs, i.e., trees].

6. Let G be a 2-connected graph. Let $e = xy$ be any edge in G. Show that $G \setminus \{e\}$ is 2-connected if and only if x and y lie on a cycle in $G \setminus \{e\}$. Conclude that a 2-connected graph is minimally 2-connected if and only if every cycle is an *induced* subgraph. [Hint: Modify the ear decomposition proof a tiny bit]