**AMS 550.472/672: Graph Theory**
**Homework Problems - Week V**

**Problems to be handed in on Wednesday, March 2:** 6, 8, 9, 11, 12.

1. **Assignment Problem.** Suppose we have a set $\{J_1, J_2, \dots, J_r\}$ of $r$ jobs to be filled by a pool of $s$ applicants $\{A_1, A_2, \dots, A_s\}$. Each job can be filed by at most one applicant and each applicant be assigned to at most one job. Also each job can be filled by only a subset of applicants qualified for the jobs. It is known in advance if a job $J_i$ can be filled by applicant $A_j$. The goal is to find the maximum number of jobs that can be filled. Formulate this as a maximum matching problem.

2. Show that in a graph $G$ whose minimum degree is $2\delta$, there is a matching of size at least $\delta$.

3. Use the matrix-tree theorem to show that the number of spanning trees in a complete graph is $n^{n-2}$.

   A **perfect matching** in a graph $G$ is a matching that covers all vertices (and thus, the graph has an even number of vertices).

4. **Structure of difference of matchings.**
   (i) Let $M, N$ be two *maximum* matchings in $G$. Describe the structure of $G' := (V(G), M\Delta N)$.
   (ii) Let $M, N$ be two perfect matchings in $G$. Describe the structure of $G' := (V(G), M\Delta N)$.
   (iii) Show that a tree can have at most one perfect matching.

5. Let $G$ be a graph with no isolated vertices. Suppose further that $G$ has a *unique* maximum matching $M$. Show that there are no $M$-alternating paths. Deduce that $M$ is a perfect matching.

6. Let $M$ be a matching in a graph $G$. Show there exists a maximum matching that covers every vertex covered by $M$. Deduce that in a graph with no isolated vertices, every vertex is covered by some maximum matching.

7. A matching $M$ is *maximal* if for every edge $e \in E(G) \setminus M$, $M \cup \{e\}$ is not a matching. In other words, there is no edge in the graph that can be added to $M$ and form a larger matching.
   (i) Give examples to show that a maximal matching need not be a maximum matching.
   (ii) Suppose $M$ is a maximal matching. Show that $|M| \geq \frac{min-vertex-cover(G)}{2}$.

8. A *line* in a matrix is a row or column of the matrix. Show that the minimum number of lines to cover all nonzero entries of a matrix (not necessarily square) is equal to the maximum number of nonzero entries, no two of which lie in a common line.

9. Let $(A_1, ..., A_p)$ and $(B_1, ..., B_q)$ be two partitions of a finite set $X$. Show that the minimum cardinality of a subset of $X$ intersecting each set among $A_1, ..., A_p, B_1, ..., B_q$ is equal to the maximum number of pairwise disjoint sets in $A_1, ..., A_p, B_1, ..., B_q$. **More precisely, show that**

$$\begin{aligned} & \min_{S \subseteq X} \{|S| : S \cap A_i \neq \emptyset,\ \forall i = 1, \dots, p, \quad \textbf{and } S \cap B_j \neq \emptyset,\ \ \forall j = 1, \dots, q\} \\ = \ & \max_{I \subseteq \{1,\dots,p\}, J \subseteq \{1,\dots,q\}} \{|I| + |J| : A_i \cap B_j = \emptyset,\ \ \forall i \in I, j \in J\} \end{aligned}$$

10. **System of distinct representatives.** Let $A_1, A_2, \dots, A_n$ be subsets of a set $X$. We say that $Y$ is a *system of distinct representatives (SDR)* for $A_1, A_2, \dots, A_n$ if there is a bijection $f : \{1, 2, \dots, n\} \to Y$ such that $f(i) \in A_i$ for every $i = 1, 2, \dots, n$.

(i) Show that a family of subsets $A_1, \ldots, A_n$ of $X$ has an *SDR* if and only if

$$|\bigcup_{i \in I} A_i| \geq |I|$$

for every subset $I \subseteq \{1, 2, \ldots, n\}$.

(ii) Let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be two partitions of the same set $X$. Show that there exists a *common SDR* for both $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ (i.e., there exists $Y \subseteq X$ such that $Y$ is an $SDR$ for both $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$) if and only if for every subset $I \subseteq \{1, 2, \ldots, n\}$, $\cup_{i \in I} A_i$ intersects at least $|I|$ sets among $B_1, B_2, \ldots, B_n$.

11. Let $G = (A \cup B, E(G))$ be a simple, bipartite graph. Prove the following generalization of Hall's matching condition: Show that

$$max - matching(G) = |A| - \max_{\emptyset \subseteq S \subseteq A} \{|S| - |N(S)|\}$$

where $N(S)$ denotes the set of neighbors of $S$. [Hint: Add some vertices and edges to the graph]

12. We say two matchings $M$ and $N$ in a graph $G$ are *disjoint* if they have no common edges, i.e., $M \cap N = \emptyset$. Let $G$ be a simple, $k$-regular (i.e., every vertex has degree $k$), bipartite graph. Show that $G$ has $k$ perfect matchings which are pairwise *disjoint* (see definition of perfect matching above Problem 4). [Hint: Use Hall's condition and induction]

13. Let $G = (A \cup B, E(G))$ be a simple, bipartite graph such that for every edge $xy$ with $x \in A$ and $y \in B$, $\deg(x) \geq \deg(y)$. Show that there exists a matching that saturates $A$.

14. Let $n$ be a fixed natural number. An $n \times n$ matrix is called a *permutation* matrix if all its entries are 0 or 1 and every row contains exactly one 1 and every column contains exactly one 1. How many $n \times n$ permutation matrices are there ? Show that a given $n \times n$ matrix with nonnegative integer entries is the sum of $k$ permutation matrices if and only if the sum in every row and every column is $k$.

15. Show that the $n$-cube graph from HW I has $n$ disjoint perfect matchings (see Problem 14). (There are at least 2 different ways to see this)

16. Let $k, r$ be natural numbers. Let $G$ be a $k$ regular, simple, bipartite graph. Show $G$ contains spanning subgraphs $G_1, G_2, \ldots G_\ell$ such that each $G_i$ is $r$-regular, and $E(G) = E(G_1) \uplus E(G_2) \uplus \ldots \uplus E(G_\ell)$ (so the edges are partitioned with no overlaps) if and only if $r$ divides $k$.

17. Do Problem 3.1.21 from the textbook.