1. **Assignment Problem.** Suppose we have a set \( \{J_1, J_2, \ldots, J_r\} \) of \( r \) jobs to be filled by a pool of \( s \) applicants \( \{A_1, A_2, \ldots, A_s\} \). Each job can be filled by at most one applicant and each applicant can be assigned to at most one job. Also each job can be filled by only a subset of applicants qualified for the jobs. It is known in advance if a job \( J_i \) can be filled by applicant \( A_j \). The goal is to find the maximum number of jobs that can be filled. Formulate this as a maximum matching problem.

Solution: Create a bipartite graph with \( r \) vertices on one side, corresponding to the jobs, and \( s \) vertices on the other side, corresponding to applicants. For each “job” vertex, put an edge from it to all applicant vertices that are qualified to do that job. Now, any matching corresponds to a valid assignment of applicants to jobs, and maximum matching corresponds to the maximum number of jobs getting filled.

2. Show that in a graph \( G \) whose minimum degree is \( 2\delta \), there is a matching of size at least \( \delta \).

Solution: By Problem 18 on HW II, we know there is a path of length \( 2\delta \). Choosing every other edge on this path, we obtain a matching of size \( \delta \).

3. Use the matrix-tree theorem to show that the number of spanning trees in a complete graph is \( n^{n-2} \).

A **perfect matching** in a graph \( G \) is a matching that covers all vertices (and thus, the graph has an even number of vertices).

4. **Structure of difference of matchings.**

   (i) Let \( M, N \) be two maximum matchings in \( G \). Describe the structure of \( G' := (V(G), M \Delta N) \).

   Solution: \( G' \) is a disjoint union of even sized cycles, even sized paths and some isolated vertices. We know the cycles are even sized from the result in class. The paths have to be even sized otherwise we have an augmenting path for one of the matchings, contradicting its maximunness.

   (ii) Let \( M, N \) be two perfect matchings in \( G \). Describe the structure of \( G' := (V(G), M \Delta N) \).

   Solution: \( G' \) is a collection of even sized cycles and isolated vertices. We cannot have a path in \( G' \) because then each of the end points of the path would not be matched in one of the matchings, contradicting the fact that we have a perfect matching.

   (iii) Show that a tree can have at most one perfect matching.

   Solution: Suppose to the contrary that we have two distinct perfect matchings \( M, N \). By part (ii), the set difference graph is a collection of cycles and isolated verities. Since \( M \) and \( N \) are distinct, we cannot be left with only isolated vertices. Therefore, we have at least one cycle contradicting the fact that it is a tree.

5. Let \( G \) be a graph with no isolated vertices. Suppose further that \( G \) has a unique maximum matching \( M \). Show that there are no \( M \)-alternating paths. Deduce that \( M \) is a perfect matching.

Solution: If there exist \( M \)-alternating paths, let \( P \) be a maximal \( M \)-alternating path. Thus, \( P \) ends at a vertex such that the edges incident on it that are not in the path are not matching edges. Then \( M \Delta P \) is a matching that is at least as large as \( M \). Since there is a unique maximum matching, this cannot happen. If we have any unmatched vertex and there are no isolated vertices, one can always have an \( M \)-alternating path starting at that vertex. Thus, no vertex is left unmatched.
6. Let $M$ be a matching in a graph $G$. Show there exists a maximum matching that covers every vertex covered by $M$. Deduce that in a graph with no isolated vertices, every vertex is covered by some maximum matching.

Solution: Let $\mathcal{N}$ be the set of all matchings that cover all the vertices covered by $M$ (and maybe more vertices). Let $N$ be a matching in $\mathcal{N}$ that is of largest size. If $N$ is not a maximum matching in $G$, there exists an $N$-augmenting path $P$. Then $N \Delta P$ is a matching with one more edge and also covers all the vertices covered by $N$. In particular, it covers all the vertices covered by $M$ and therefore $N \Delta P$ is in $\mathcal{N}$. This contradicts the choice of $N$.

7. A matching $M$ is maximal if for every edge $e \in E(G) \setminus M$, $M \cup \{e\}$ is not a matching. In other words, there is no edge in the graph that can be added to $M$ and form a matching.

(i) Give examples to show that a maximal matching need not be a maximum matching.

Solution: Consider the following graph:

The edge 24 is maximal matching but not a maximum matching which is of size 2.

(ii) Suppose $M$ is a maximal matching. Show that $|M| \geq \frac{\min \text{ vertex cover}(G)}{2}$.

Solution: Consider a maximum matching $N$. If there exists an edge $e$ in $N$ such that both end points are left unmatched by $M$, then $M \cup \{e\}$ is a matching, contradicting the maximality of $M$. Thus, the set $S$ of endpoints of the edges in $M$ form a vertex cover. Therefore, $2 \times |M| = |S| \geq \min \text{ vertex cover}(G)$. The result follows by divine through by 2.

8. A line in a matrix is a row or column of the matrix. Show that the minimum number of lines to cover all nonzero entries of a matrix is equal to the maximum number of nonzero entries, no two of which lie in a common line.

Solution: Consider a bipartite graph $G = (A \cup B, E(G))$ where $A$ corresponds to the rows of the matrix, and $B$ corresponds to the columns and there is an edge between two vertices if the corresponding entry in the matrix is nonzero. The minimum number of lines to cover all nonzero entries is thus a minimum vertex cover. The maximum number of non zero entries no two of which are in a common line forms a set of matching edges. The result now follows from König’s theorem for bipartite graphs.

9. Let $(A_1, \ldots, A_p)$ and $(B_1, \ldots, B_q)$ be two partitions of the finite set $X$. Show that the minimum cardinality of a subset of $X$ intersecting each set among $A_1, \ldots, A_p, B_1, \ldots, B_q$ is equal to the maximum number of pairwise disjoint sets in $A_1, \ldots, A_p, B_1, \ldots, B_q$. More precisely, show that

$$\min_{S \subseteq X} \{|S| : S \cap A_i \neq \emptyset, \forall i = 1, \ldots, p, \text{ and } S \cap B_j \neq \emptyset, \forall j = 1, \ldots, q\} = \max_{I \subseteq \{1, \ldots, p\}, J \subseteq \{1, \ldots, q\}} \{|I| + |J| : A_i \cap B_j = \emptyset, \forall i \in I, j \in J\}$$
Solution: Construct a bipartite graph \( G = (A \cup B, E(G)) \) where \( A = \{1, \ldots, p\} \) and \( B = \{1, \ldots, q\} \). For every element \( x \in X \), there is a unique \( i \) and \( j \) such that \( x \in A_i \) and \( x \in B_j \) (since we have a partition). Corresponding to this element, we put an edge between \( i \) and \( j \) in \( G \). Now the minimum cardinality of a subset of \( X \) intersecting each set among \( A_1, \ldots, A_p, B_1, \ldots, B_q \) corresponds to a minimum edge cover, and the maximum number of pairwise disjoint sets in \( A_1, \ldots, A_p, B_1, \ldots, B_q \) is a maximum independent set. So the result follows from the consequence of Gallai’s theorem and König’s theorem that in a bipartite graph, minimum edge cover equals maximum independent set.

10. **System of distinct representatives.** Let \( A_1, A_2, \ldots, A_n \) be subsets of a set \( X \). We say that \( Y \) is a system of distinct representatives (SDR) for \( A_1, A_2, \ldots, A_n \) if there is a bijection \( f : \{1, 2, \ldots, n\} \to Y \) such that \( f(i) \in A_i \) for every \( i = 1, 2, \ldots, n \).

   (i) Show that a family of subsets \( A_1, \ldots, A_n \) of \( X \) has an SDR if and only if
   \[
   \big| \bigcup_{i \in I} A_i \big| \geq |I|
   \]
   for every subset \( I \subseteq \{1, 2, \ldots, n\} \).

   Solution: Construct a bipartite graph \( G = (A \cup B, E(G)) \) where \( A = \{1, \ldots, n\} \) and \( B = X \). We put edges between \( i \in A \) and \( x \in A_i \) if \( x \in X \). An SDR is then the same thing as a matching that saturates \( A \). The condition stated above is now simply a statement of Hall’s matching condition.

   (ii) Let \( A_1, A_2, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \) be two partitions of the same set \( X \). Show that there exists a common SDR for both \( A_1, A_2, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \) (i.e., there exists \( Y \subseteq X \) such that \( Y \) is an SDR for both \( A_1, A_2, \ldots, A_n \) and \( B_1, B_2, \ldots, B_n \) if and only if for every subset \( I \subseteq \{1, 2, \ldots, n\} \), \( \cup_{i \in I} A_i \) intersects at least \( |I| \) sets among \( B_1, B_2, \ldots, B_n \).

   Solution: We use the same bipartite graph as constructed in Problem 9 above. A common SDR is simply a perfect matching in this graph, and the condition stated above is equivalent to Hall’s matching condition.

11. Let \( G = (A \cup B, E(G)) \) be a simple, bipartite graph. Prove the following generalization of Hall’s matching condition: Show that
   \[
   \nu(G) = |A| - \max_{\emptyset \subseteq S \subseteq A} \{|S| - |N(S)|\}
   \]
   where \( N(S) \) denotes the set of neighbors of \( S \). [Hint: Add some vertices and edges to the graph]

   Solution: Let \( k = \max_{\emptyset \subseteq S \subseteq A} \{|S| - |N(S)|\} \). Add \( k \) vertices in \( B \) and connect these new vertices to every vertex in \( A \). Call the new graph \( G' \). Let the new set of \( k \) vertices be called \( K \). Now, Hall’s condition is satisfied: For any \( \emptyset \subseteq S \subseteq A \), the neighborhood \( N_{G'}(S) \) of \( S \) in \( G' \) is simply \( N_G(S) \cup K \) where \( N_G(S) \) is the neighborhood of \( S \) in \( G \). So \( |N_{G'}(S)| = |N_G(S)| + k \geq |N_G(S)| + (|S| - |N_G(S)|) = |S| \). So there is a matching in \( G' \) that saturates \( A \). Now removing the matching edges that are incident on \( K \), we obtain a matching in \( G \). But we remove at most \( k \) matching edges when we remove \( K \), so we obtain a matching of size at least \( |A| - k \). Notice that any maximum matching has size at most \( |A| - k \). So this is a maximum matching.

12. We say two matchings \( M \) and \( N \) in a graph \( G \) are **disjoint** if they have no common edges, i.e., \( M \cap N = \emptyset \). Let \( G \) be a simple, \( k \)-regular (i.e., every vertex has degree \( k \)), bipartite graph.
Show that $G$ has $k$ perfect matchings which are pairwise disjoint (see definition of perfect matching above Problem 4). [Hint: Use Hall’s condition and induction]

Solution: We first prove that in a $k$-regular bipartite graph, there always exists a perfect matching. We check Hall’s condition: Consider any $S \subseteq V_1$. The number of edges incident on $S$ equals $k|S|$ since each vertex has degree $k$. All these edges are incident on the neighborhood $N(S)$. Also, $N(S)$ may have edges coming into it from vertices outside $S$. Therefore, it has at least $k|S|$ edges incident on it. But the number of edges incident on $N(S)$ is $k|N(S)|$. Therefore, $k|S| \leq k|N(S)|$ implying that $|S| \leq |N(S)|$.

We prove it by induction on $k$. If $k = 1$, then every vertex has degree one, and we have a perfect matching, so we are done. For the induction step, from a corollary of Hall’s theorem, every regular bipartite graph has a perfect matching. Removing these matching edges leaves a $k - 1$-regular graph, which by induction has $k - 1$ disjoint perfect matchings. Together with the first matching edges, we obtain $k$ disjoint perfect matchings.

14. Let $n$ be a fixed natural number. An $n \times n$ matrix is called a permutation matrix if all its entries are 0 or 1 and every row contains exactly one 1 and every column contains exactly one 1. How many $n \times n$ permutation matrices are there? Show that a given $n \times n$ matrix with nonnegative integer entries is the sum of $k$ permutation matrices if and only if the sum in every row and every column is $k$.

Solution: There are $n!$ permutation matrices, corresponding to every permutation. We model an $n \times n$ matrix $M$ with nonnegative integer entries as a bipartite graph (not necessarily simple) with the rows as one side of the vertices, and the columns as the other side. For any two vertices $i, j$, we put $M_{ij}$ edges between $i$ and $j$. Each vertex now has degree $k$ because of the condition that the row sums and the columns sums are all equal to $k$. Using Problem 5 above, we have $k$ disjoint perfect matchings in this bipartite graph. Finally observe that each such perfect matching corresponds to a permutation matrix and the sum of these permutation matrices will give the original matrix $M$.

15. Show that the $n$-cube graph from HW I has $n$ disjoint perfect matchings (see Problem 14). (There are at least 2 different ways to see this)

Solution: We know from HW II, Problem 7, the $n$-cube graph is bipartite. Moreover, every vertex has degree $n$. Thus, by Problem 12 above, we have the result. [One can also explicitly construct the $n$ disjoint perfect matchings: Fix a coordinate and match every vertex to its partner on this coordinate. Another way to do this by induction on $n$ and using the fact that the $n$-cube graph can be obtained by taking two copies of the $n - 1$-cube and connecting corresponding vertices in the two copies with an edge.]
16. Let $k, r$ be natural numbers. Let $G$ be a $k$ regular, simple, bipartite graph. Show $G$ contains spanning subgraphs $G_1, G_2, \ldots G_\ell$ such that each $G_i$ is $r$-regular, and $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_\ell)$ (so the edges are partitioned with no overlaps) if and only if $r$ divides $k$.

Solution: $(\Rightarrow)$ In this case, we have $k = \ell \cdot r$ and so we are done.

$(\Leftarrow)$ By Problem 5 above, we have $k$ disjoint perfect matchings. Putting $r$ of these together yields a $r$-regular bipartite subgraph. Since $r$ divides $k$, we can group $r$ of these matchings together and have $k/r$ such $r$-regular subgraphs.


Solution for 3.1.21: Let $e = xy$ be any edge with $x \in X, y \in Y$. Consider the bipartite graph $G' = G \setminus \{x, y\}$. Let $X' = X \setminus \{x\}, Y' = Y \setminus \{y\}$. For any subset $S \subseteq X'$, $N_{G'}(S) = N_G(S) \setminus \{y\}$. Thus, $|N_{G'}(S)| \geq |N_G(S)| - 1$ (equality holds iff $y \in N_G(S)$). Since $|N_G(S)| > |S|$, we have $|N_{G'}(S)| \geq |S|$ and thus, we have a matching that saturates $X'$. Adding the edge $e$ to this matching, we have a matching that saturates $X$ that contains $e$. 