AMS 550.472/672: Graph Theory
Homework Problems - Week IV

Problems to be handed in on Wednesday, Feb 24: 4, 5, 8.

1. Recall that a tree is always bipartite. Show that a tree always has a leaf in its larger partite set.

2. Let the maximum degree of a vertex in a tree be Δ. Show that there are at least Δ leaves in the tree.

3. Let T be a tree such that every leaf is adjacent to a vertex of degree at least 3. Show that there are two leaves with a common neighbor.

4. Let $d_1, d_2, \ldots, d_n$ be n strictly positive integers with $n \geq 2$. Show that there exists a tree with vertex degrees $d_1, d_2, \ldots, d_n$ if and only if $d_1 + d_2 + \ldots + d_n = 2n - 2$.

5. Show that a tree with no vertex of degree 2, has more leaves than non-leaf vertices.

6. **Oriented Trees (Arborescences).** Let T be a undirected tree and let $\hat{T}$ be an orientation of T such that every vertex is the head of at most one edge. Such an oriented tree is called an *arborescence*.

   (i) Show that there exists a vertex that is not the head for any edge. This vertex is called a *root* of $\hat{T}$.

   (ii) Show that for every vertex there is a unique directed path to it from a root. Thus conclude that $\hat{T}$ has a unique root.

7. Let T be a tree with $2k$ vertices of odd degree. Prove that we can find k paths $P_1, P_2, \ldots, P_k$ such that $E(T) = E(P_1) \uplus E(P_2) \uplus \ldots \uplus E(P_k)$.

8. Common Intersection property. Let G be a simple graph. We say that a finite collection of subgraphs $G_1, G_2, \ldots, G_k$ has the *common intersection property* if the following logical implication is true:

$$(\forall i \neq j \;\; V(G_i) \cap V(G_j) \neq \emptyset) \Longrightarrow (V(G_1) \cap V(G_2) \cap \ldots \cap V(G_k) \neq \emptyset).$$

   [In other words, if we have a common vertex for every pair of subgraphs, then there is a common vertex for the whole family.]

   (a) Suppose T is a tree, and $T_1, T_2, \ldots, T_k$ are subgraphs of T that are also trees (but not necessarily spanning). Show that $T_1, \ldots, T_k$ have the common intersection property.

   (b) Show that a simple graph G is a forest if and only if every collection of subpaths (subgraphs that are paths) $G_1, \ldots, G_k$ has the common intersection property.
   [Hint: Induction is your best friend!]

9. Let G be a simple n-vertex graph having $n - 2$ edges. Show that either G has an isolated vertex, or has two components each of which is a tree with at least 2 vertices.

10. **Simultaneous exchange property.** Let T, T' be two distinct spanning trees of a simple graph G. For any $e \in E(T) \setminus E(T')$, show that there exists $e' \in E(T') \setminus E(T)$ such that both $(T \setminus \{e\}) \cup \{e'\}$ and $(T' \setminus \{e'\}) \cup \{e\}$ are spanning trees of G.