AMS 550.472/672: Graph Theory
Homework Problems - Week III

Problems to be handed in on Wednesday, Feb 17: 2, 7, 10, 11(a).

1. **Girth of a graph.** We define *girth* of a graph G as the length of the shortest cycle in G.
   (a) Let G be a simple graph with girth 4 (these graphs are called *triangle-free*). Further, suppose that every vertex has degree k, where k is some natural number. Show that G has at least $2k$ vertices.
   (b) Let G be a simple graph with girth 5 (these graphs are called *quadrilateral-free*). Further, suppose every vertex has degree k, where k is some natural number. Show that G has at least $k^2 + 1$ vertices.

2. Let G be a connected simple graph with an even number of edges, so $|E(G)| = 2k$ for some natural number k. Show that $E(G)$ can be partitioned as $E(G) = E(P_1) \uplus E(P_2) \uplus \ldots \uplus E(P_k)$ where $P_i \subseteq G$ is a path of length two for each $i = 1, \ldots, k$. [If you need a hint, please email me].

3. Let $G = (V(G), E(G))$ be a simple bipartite graph with bipartition $V(G) = V_1 \cup V_2$. Let $n = |V(G)|$, $r = |V_1|$, $s = |V_2|$ and $m = |E(G)|$. Show that $m \leq rs$. Deduce that $m \leq \frac{n^2}{4}$ and describe the graphs where this holds at equality.

4. Let G be a simple triangle-free graph, such that every pair of non-adjacent vertices have *exactly* two common neighbors. Show that G is regular, i.e., every vertex has the same degree.

5. Show that for any directed graph $G = (V(G), E(G))$, $\sum_{v \in V(G)} d^+(v) = |E(G)| = \sum_{v \in V(G)} d^-(v)$.

6. Prove that there exists an n-vertex tournament with in-degree equal to out-degree for every vertex iff n is odd.

7. Show that in any tournament, there is a directed path which visits every vertex.

8. **Directed acyclic graphs.** A directed graph which has no directed cycles is called a *directed acyclic graph (DAG)*. Note that the underlying undirected graph may have cycles.
   (i) Show that in any directed acyclic graph, there is a vertex whose in-degree equals 0, i.e., it is not the head for any edge. Such a vertex is called a *source*.
   (ii) Show that in any directed acyclic graph, there is a vertex whose out-degree equals 0, i.e., it is not the tail for any edge. Such a vertex is called a *sink*.
   (iii) Show that in any directed acyclic graph, one can order the vertices so as to respect edge directions: i.e., show there exists a one-to-one and onto mapping $f\colon V(G) \to \{1, \ldots, n\}$ such that for every directed edge (u, v), $f(u) \leq f(v)$. So every edges points from a lower numbered vertex to a higher numbered vertex. This kind of an ordering is called a *topological sort* of the vertices of a DAG.
   (iv) We say that a partition of the vertices $V = L_0 \uplus L_1 \uplus \ldots \uplus L_{k-1}$ is a *stratification with k levels*, if every directed edge e is between vertices in different levels, and the edge points from a lower indexed level to a higher indexed level. Notice that the topological sorting from part (iii) is a stratification with $|V|$ levels. Let G_0 be a directed acyclic graph and let $\hat{L}_0$ be the set of sources in G_0. Consider the following graphs. For each $i = 1, 2, \ldots$, define $G_i = G_{i-1} \setminus L_{i-1}$ and L_i is the set of sources in G_i. Notice that for some natural

number k^*, for every $i \geq k$, G_i is just the empty graph, i.e., empty set of vertices and empty set of edges. Show that $V = L_0 \uplus L_1 \uplus \ldots \uplus L_{k^*-1}$ is a stratification.

Bonus question: Show that k^* above is the smallest value of $k \in \mathbb{N}$ such that there exists a stratification with k levels.

9. Show that a forest with k edges has $n - k$ connected components.

10. Let G be a connected n-vertex graph. Show that G has exactly one cycle if and only if G has exactly n edges.

11. **Cut Vertex in a graph.** A vertex $v \in V(G)$ is a *cut-vertex* of G, if $G \setminus \{v\}$ has more connected components than G.

    (a) Show that every graph has at least 2 vertices that are NOT cut vertices.

    (b) Suppose G is a simple connected graph with exactly two vertices that are not cut vertices, then show that G is a path.