AMS 550.472/672: Graph Theory
Homework Problems - Week II

Problems to be handed in on Wednesday, Feb 10: 3, 8, 10, 14, 15.

1. Show that the n-CUBE Q_n and the *Boolean lattice* BL_n (see HW 1, Problems 4 and 5 for the definitions of these graphs) are connected for every natural number n.

2. Let W be a walk of length at least 1 whose first and last endpoints are the same. Moreover, suppose W does not contain a cycle. Show that some edge of W repeats immediately (once in each direction).

3. Let G be a simple graph with n vertices and m edges. Show that if $m > \binom{n-1}{2}$, then G is connected. For every $n > 1$, find a disconnected simple graph G with $m = \binom{n-1}{2}$.

4. Let G and H be isomorphic graphs. Show that G is connected if and only if H is connected.

5. Let G and H be isomorphic graphs. Show that G is bipartite if and only if H is bipartite.

6. Do problems 1.2.18, 1.2.19, 1.2.22 from the textbook.

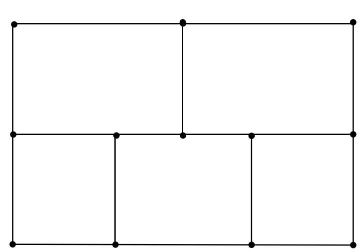
7. Show that the n-CUBE graph Q_n and the boolean lattice BL_n are bipartite for every natural number n.

8. Let G_n be a graph whose vertex set are all $n!$ permutations of the set $\{1, 2, \ldots, n\}$ (e.g., for $n = 3$, G_n has 6 vertices: {123, 132, 213, 231, 312, 321}.) Two vertices (permutations) are adjacent in G_n if and only if they differ by interchanging a pair of entries (e.g., 123 and 132 are adjacent, but 123 and 312 are NOT adjacent in the above $n = 3$ example. Another example: 1234 and 1432 are adjacent, but 1234 and 4321 are NOT adjacent in the $n = 4$ case.) Is G_n connected ? Is it bipartite ?

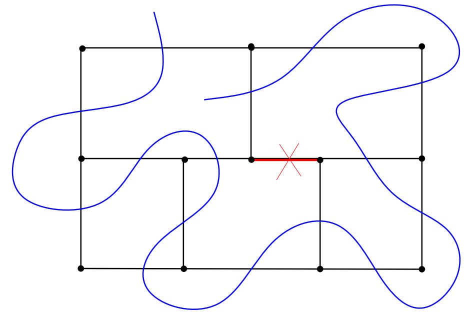
9. Let $G = (V(G), E(G))$ be a simple bipartite graph with bipartition $V(G) = V_1 \cup V_2$. Let $n = |V(G)|$, $r = |V_1|$, $s = |V_2|$ and $m = |E(G)|$. Show that $m \leq rs$. Deduce that $m \leq \frac{n^2}{4}$ and describe the graphs where this holds at equality.

10. What is the maximum size independent set in a cycle of length n when (a) n is even, (b) n is odd ? Use this to show that a graph G is bipartite if and only if every subgraph G' has an independent set of size at least $\frac{|V(G')|}{2}$.

11. Consider the following blueprint for a house. Each box represents a room, and the lines connecting two dots denote walls separating adjoining rooms (so there are 16 walls in all).



The problem is to draw a curve in the plane that passes through each wall exactly once. The curve may start at any point in the plane and end at any other point (the start and end points don't have to be the same). Here is an example curve that misses a wall.



Also, one cannot "kill two walls with one pass", meaning that you cannot go through a corner and claim that you covered two (or three) walls at the same time. Either find such a curve or show that none exists.

12. **Complement of a graph.** Let $G = (V(G), E(G))$ be a simple graph. Then the complement of G, denoted by $\bar{G}$, is the simple graph with $V(\bar{G}) = V(G)$ and $uv \in E(\bar{G}) \Leftrightarrow uv \notin E(G)$.

    (a) Show that G and H are isomorphic if and only if $\bar{G}$ and $\bar{H}$ are isomorphic.

    (b) Show that for any disconnected simple graph G, $\bar{G}$ is connected.

    (c) Let G be a simple graph on 6 vertices. Show that either G or $\bar{G}$ contains a triangle, i.e., a subset of three vertices all of which are connected to each other.

13. Let $k \in \mathbb{N}$. Let G be a graph with n vertices and suppose every vertex in a graph has degree k. How many edges does the graph have?

14. Recall the n-CUBE graph Q_n from HW for Week I. How many vertices and edges does Q_n have? How many vertices and edges does the boolean lattice BL_n have?

15. Suppose u and v are the only vertices in G that have odd degree. Is it true that G contains a $u - v$ path ? (Careful: G is not assumed to be connected.)

16. Do Problems 1.3.2 and 1.3.9 from the textbook.

17. **Even Graphs.** We call a graph *even* if every vertex has even degree. Prove that the number of simple even graphs on n vertices is $2^{\binom{n-1}{2}}$.

18. Let G be a simple graph, where the minimum degree of a vertex is k. Show that G contains a path of length at least k and a cycle of length at least $k + 1$.