AMS 550.472/672: Graph Theory
Homework Problems - Week XIII

1. Let $k \in \mathbb{N}$ be a fixed natural number. Recall that the Ramsey number $R(k)$ is the smallest natural number n such that every graph on n vertices contains K_k or $\overline{K_k}$. Show that for every $N \in \mathbb{N}$, $R(k) > N - \binom{N}{k} 2^{1-\binom{k}{2}}$. (Using the right choice of N, this can be used to show that $R(k) > \frac{k}{e} 2^{k/2}$ which is a slight improvement over the bound we saw in class) [Hint: Use expectation to compute the number of "bad" subgraphs and then remove vertices to get rid of these "bad" subgraphs]

2. Let $p(n)$ be a fixed function (could be the constant function). Suppose $\mathcal{P}_1$ is a graph property that holds for almost all graphs in $\mathcal{G}(n,p)$, and $\mathcal{P}_2$ be another graph property that holds for almost all graphs in $\mathcal{G}(n,p)$.

   (i) Show that $\mathcal{P}_1 \cap \mathcal{P}_2$ is also a graph property.

   (ii) Show that the property $\mathcal{P}_1 \cap \mathcal{P}_2$ holds for almost all graphs.

3. Let k be a fixed natural number and $0 < p < 1$ be a constant. Show that almost every graph in $\mathcal{G}(n,p)$ is k-connected. [Hint: Show that, in fact, for almost every graph, all pairs of vertices have k paths of length two connecting them.]

4. Let $\epsilon > 0$ and let $0 < p(n) < 1$ be a function of $n \in \mathbb{N}$, and let $r(n)$ be an integer valued function of n such that $r(n) \geq (1+\epsilon)\frac{2\ln n}{p(n)}$ for all $n \in \mathbb{N}$. Show that almost no graph in $\mathcal{G}(n,p)$ contains $r(n)$ independent vertices.

5. Do 8.5.26.

6. Let $0 < \epsilon \leq 1$ be a constant and $p(n) = (1-\epsilon)(\ln n)\frac{1}{n}$.

   (i) Show that $n(1-p)^{n-1} \to \infty$ as $n \to \infty$.

   (ii) Show that almost every graph in $\mathcal{G}(n,p)$ contains an isolated vertex.

   (iii) Find a function $m(n)$ such that almost every graph in $\mathcal{G}(n,p)$ has at least $m(n)$ isolated vertices.

7. Find a probability function $p(n)$ such that almost every graph in $\mathcal{G}(n,p)$ is disconnected, but the expected number of spanning trees of G tends to infinity as $n \to \infty$. (This concretely shows an example where it is impossible to lower bound the probability $P(X \geq 1)$ by simply lower bounding $E[X]$ - Markov doesn't work the other way)

8. Show that if $np(n) \to 0$ as $n \to \infty$ then almost every graph in $\mathcal{G}(n,p)$ is a forest. [Hint: Markov]

9. Let $k \in \mathbb{N}$ be fixed. Does the property of containing *any* tree on k vertices have a threshold function in $\mathcal{G}(n,p)$ (we are not considering a fixed tree, as discussed in class) ? If so, which one ? If not, why not ?

10. Let $k \in \mathbb{N}$ be fixed. Determine the threshold function for containing the k-cube Q_k in $\mathcal{G}(n,p)$. (see Problem 4 on HW I).