1. Let \( k \in \mathbb{N} \) be a fixed natural number. Recall that the Ramsey number \( R(k) \) is the smallest natural number \( n \) such that every graph on \( n \) vertices contains \( K_k \) or \( \overline{K_k} \). Show that for every \( N \in \mathbb{N} \), \( R(k) > N - \left( \frac{N}{k} \right) 2^{1-\left(\frac{k}{2}\right)} \). (Using the right choice of \( N \), this can be used to show that \( R(k) > \frac{k}{2} 2^{k/2} \) which is a slight improvement over the bound we saw in class) [Hint: Use expectation to compute the number of “bad” subgraphs and then remove vertices to get rid of these “bad” subgraphs]

Solution: Let \( X \) be the random variable that counts the number of copies of \( K_k \) and \( \overline{K_k} \) in \( G \in \mathcal{G}(N, \frac{1}{2}) \). Using linearity of expectation, we obtain that \( E[X] = \left( \frac{N}{k} \right) 2^{1-\left(\frac{k}{2}\right)} \). Thus, there exists a graph on \( N \) vertices that has at most \( \left( \frac{N}{k} \right) 2^{1-\left(\frac{k}{2}\right)} \) copies of \( K_k \) or \( \overline{K_k} \). For each of these copies, remove a vertex: so we remove at most \( \left( \frac{N}{k} \right) 2^{1-\left(\frac{k}{2}\right)} \) vertices and are left with at least \( N - \left( \frac{N}{k} \right) 2^{1-\left(\frac{k}{2}\right)} \) vertices. This graph has no copies of \( K_k \) or \( \overline{K_k} \) because we destroyed all copies by removing a vertex from the subgraph. Thus, \( R(k) > N - \left( \frac{N}{k} \right) 2^{1-\left(\frac{k}{2}\right)} \).

2. Let \( p(n) \) be a fixed function (could be the constant function). Suppose \( \mathcal{P}_1 \) is a graph property that holds for almost all graphs in \( \mathcal{G}(n,p) \), and \( \mathcal{P}_2 \) be another graph property that holds for almost all graphs in \( \mathcal{G}(n,p) \).

(i) Show that \( \mathcal{P}_1 \cap \mathcal{P}_2 \) is also a graph property.

Solution: Consider \( G \in \mathcal{P}_1 \cap \mathcal{P}_2 \), and any \( H \) isomorphic to \( G \). We need to show that \( H \in \mathcal{P}_1 \cap \mathcal{P}_2 \). Since \( \mathcal{P}_1 \) is a property, and \( G \in \mathcal{P}_1 \) we have that \( H \in \mathcal{P}_1 \). Similarly, \( H \in \mathcal{P}_2 \) and so \( H \in \mathcal{P}_1 \cap \mathcal{P}_2 \).

(ii) Show that the property \( \mathcal{P}_1 \cap \mathcal{P}_2 \) holds for almost all graphs.

Solution: We show that for any \( \epsilon > 0 \), there exists \( n \in \mathbb{N} \) such that \( P_{\mathcal{G}(n,p)}(G \in \mathcal{P}_1 \cap \mathcal{P}_2) \geq 1 - \epsilon \). For both \( i = 1, 2 \), since \( \mathcal{P}_i \) is a graph property that holds for almost all graphs in \( \mathcal{G}(n,p) \), there exists \( n_i \in \mathbb{N} \) such that \( P_{\mathcal{G}(n,p)}(G \in \mathcal{P}_i) > 1 - \frac{\epsilon}{2} \). Therefore, for \( n \geq \max\{n_1, n_2\} \), we have \( P_{\mathcal{G}(n,p)}(G \in \mathcal{P}_1 \cap \mathcal{P}_2) = P_{\mathcal{G}(n,p)}(G \in \mathcal{P}_1 \cup \mathcal{P}_2) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \). Thus, \( P_{\mathcal{G}(n,p)}(G \in \mathcal{P}_1 \cap \mathcal{P}_2) = 1 - P_{\mathcal{G}(n,p)}(G \in \mathcal{P}_1 \cup \mathcal{P}_2) \geq 1 - \epsilon \).

3. Let \( k \) be a fixed natural number and \( 0 < p < 1 \) be a constant. Show that almost every graph in \( \mathcal{G}(n,p) \) is \( k \)-connected. [Hint: Show that, in fact, for almost every graph, all pairs of vertices have \( k \) paths of length two connecting them.]

Solution: Let \( X \) be the random variable that counts the number of pair of vertices that do not have \( k \) length two paths connecting them. We will compute the expectation of \( X \); for this purpose we define the indicator random variable \( X_{uv} \) for every pair of vertices \( u, v \) that takes value 1 if and only if there are at most \( k - 1 \) length two paths connecting \( u \) and \( v \). Thus, \( E[X] = \sum_{u,v \in V(G)} E[X_{uv}] \).

For any \( k \) subset \( S \subseteq V(G) \setminus \{u, v\} \) define the event \( A_S \) as the set of all graphs in \( \mathcal{G}(n,p) \) such that the path \( uv \) exists in \( G \) for every \( x \in S \). Further, let \( S_1, \ldots, S_{(n-2)/k} \) be fixed \( k \)-subsets that are disjoint; i.e., \( S_i \cap S_j = \emptyset \) and \( x, y \notin S_i \). Then

\[
P[X_{uv} = 1] = P[\bigcap_{S \subseteq V(G) \setminus \{u, v\}} \bigcap_{S \subseteq V(G) \setminus \{u, v\}} \bigcap_{S \subseteq (n-2)/k}] = P[A_{S_1}] \cdot P[A_{S_2}] \cdot \ldots \cdot P[A_{S_{(n-2)/k}}].
\]

Finally, observe that \( P[A_{S_i}] = (1 - p^{2k}) \). So \( P[X_{uv} = 1] \leq (1 - p^{2k})^{(n-2)/k} \).

Therefore, \( E[X] = \sum_{u,v \in V(G)} E[X_{uv}] = \sum_{u,v \in V(G)} P[X_{uv} = 1] \leq n^2 (1 - p^{2k})^{(n-2)/k} \). Since \( 1 - p^{2k} \) is a constant strictly less than 1, and the exponential function decays faster than
any polynomial function, \( n^2(1 - p^{2k})((n-2)/k) \rightarrow 0 \) as \( n \rightarrow \infty \); showing that \( E[X] \rightarrow 0 \). By Markov’s inequality, we have that \( P[X \geq 1] \leq E[X] \rightarrow 0 \). Therefore, \( P[X = 0] \rightarrow 1 \) as \( n \rightarrow \infty \). \( X = 0 \) means for every pair of vertices, there exist \( k \) length two paths: by Menger’s theorem, this shows that \( G \) is \( k \)-connected.

4. Let \( \epsilon > 0 \) and let \( 0 < p(n) < 1 \) be a function of \( n \in \mathbb{N} \), and let \( r(n) \) be an integer valued function of \( n \) such that \( r(n) \geq (1 + \epsilon)^{2\ln n/p(n)} \) for all \( n \in \mathbb{N} \). Show that almost no graph in \( G(n, p) \) contains \( r(n) \) independent vertices.

Solution: Let \( X \) be the random variable that counts the number of copies of \( \overline{K}_r \) in \( G \in G(n, p) \). Using the same analysis as in Problem 5 from HW 11, we see that \( E[X] = \binom{n}{r}(1-p)^{\binom{r}{2}} \). Also, observe that the probability that a graph in \( G(n, p) \) has \( r \) independent vertices is the same as the probability that \( X \geq 1 \). We will show that \( E[X] \rightarrow 0 \) as \( n \rightarrow \infty \). By Markov, this implies that \( P(X \geq 1) \leq E[X] \rightarrow 0 \) and we will be done.

\[
E[X] = \binom{n}{r}(1-p)^{\binom{r}{2}} \\
\leq n^r(1-p)^{r(r-1)/2} \\
\leq (ne^{-p(r-1)/2})^r \quad \text{using } e^{-p} \geq 1 - p \\
\leq (ne^{-p r^2 / 2})^r \\
\leq (ne^{(1+\epsilon)\ln n \sqrt{e}})^r \quad \text{using } r(n)p(n) \geq (1 + \epsilon)2 \ln n \text{ and } p \leq 1 \\
\leq (\frac{n^r}{e^r})^r \\
\leq (\frac{n^r}{e^r})(1+\epsilon)^{2\ln n} \quad \text{using } r(n) \geq r(n)p(n) \geq (1 + \epsilon)2 \ln n \text{ since } p(n) \leq 1
\]

The term inside the exponent goes to 0 as \( n \rightarrow \infty \), and the exponent goes to \( \infty \). Thus, the expression goes to 0 as desired.


6. Let \( 0 < \epsilon \leq 1 \) be a constant and \( p(n) = (1-\epsilon)(\ln n)^{\frac{1}{n}} \).

(i) Show that \( n(1-p)^{n-1} \rightarrow \infty \) as \( n \rightarrow \infty \).

Solution:
\[
n(1-p)^{n-1} \geq n(1-p)^n = n(1-(1-\epsilon)(\ln n)^{\frac{1}{n}})^n = n(1 + (\ln n(\epsilon^{-1}))^{\frac{1}{n}})^n
\]

Recall that \( \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e \). Therefore, \( \lim_{n \rightarrow \infty} (1 + (\ln n(\epsilon^{-1}))^{\frac{1}{n}})^n = \lim_{n \rightarrow \infty} e^{\ln n(\epsilon^{-1})} = \lim_{n \rightarrow \infty} n(\epsilon^{-1}) \). Thus, \( \lim_{n \rightarrow \infty} n(1 + (\ln n(\epsilon^{-1}))^{\frac{1}{n}})^n = \lim_{n \rightarrow \infty} n^\epsilon = \infty \).

(ii) Show that almost every graph in \( G(n, p) \) contains an isolated vertex.

Solution: Let \( X \) be the random variable which counts the number of isolated vertices in \( G(n, p) \). For each vertex \( v \in \{1, \ldots, n\} \), let \( X_v \) be the random variable which takes value 1 if and only if \( v \) is isolated. We compute \( \mu := E[X] = \sum_v E[X_v] = \sum_v (1-p)^{n-1} = n(1-p)^{n-1} \). We also compute \( E[X^2] = \sum_{u,v} E[X_u X_v] = \sum_u E[X_u^2] + \sum_{u \neq v} E[X_u X_v] = n(1-p)^{n-1} + \binom{n}{2}(1-p)^{2n-3} \). Therefore,
\[
\sigma^2 = \frac{n(1-p)^{n-1} + \binom{n}{2}(1-p)^{2n-3} - \mu^2}{\mu^2} \leq \frac{n(1-p)^{n-1} + n^2(1-p)^{2n-3}}{\mu^2} - 1
\]

Substituting \( \mu = n(1-p)^{n-1} \), the above expression becomes \( \frac{1}{n(1-p)^{n-1}} + \frac{n^2(1-p)^{2n-3}}{n^2(1-p)^{2n-2}} - 1 \).

Thus we get
\[
\frac{\sigma^2}{\mu^2} \leq \frac{1}{n(1-p)^{n-1}} + \frac{1}{1-p} - 1
\]
Let $k$ be fixed. Determine the threshold function for containing the $k$-dimensional cube $Q_k$ in $G(n,p)$. (See Problem 10 on HW 1).

Solution: $Q_k$ is a $k$-regular graph. Thus, $\epsilon(Q_k) = k/2$. If we remove edges, then we only decrease the ratio of the edges to vertices. If we remove vertices, we decrease some of the degrees, and so the ratio of the edges to vertices decreases, because this is equal to ratio of the sum of the degrees and 2 times the number of vertices. Thus, $\epsilon'(Q_k) = k/2$. Therefore, the threshold function for containing a copy of $Q_k$ is $(1/n)^{2/k}$. 

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By part (i), the first term goes to 0 and observe that $p(n) \to 0$ as $n \to \infty$ and so $\frac{1}{1-p} \to 1$ as $n \to \infty$. Thus, $\frac{\sigma^2}{n^2} \to 0$

(iii) Find a function $m(n)$ such that almost every graph in $G(n,p)$ has at least $m(n)$ isolated vertices.

Solution: This problem is canceled - it uses certain asymptotic analyses which we haven’t discussed in class.

7. Find a probability function $p(n)$ such that almost every graph in $G(n,p)$ is disconnected, but the expected number of spanning trees of $G$ tends to infinity as $n \to \infty$. (This concretely shows an example where it is impossible to lower bound the probability $P(X \geq 1)$ by simply lower bounding $E[X]$ - Markov doesn’t work the other way)

Solution: Let $p(n)$ be the probability function from Problem 6(ii) above. By Problem 8 on “HW for Week XII”, we have that $E[X] = n^{n-2}p^{n-1}$ where $X$ is the random variable that counts the number of spanning trees. From Problem 6(ii) above, we know that almost every graph has an isolated vertex which implies the graph is disconnected. However, $E[X] \to \infty$: $E[X] = \frac{np^{n-1} - 1}{n}, np = (1-\epsilon)\ln n$; thus $(np)^{n-1} = (1-\epsilon)\ln n) / n \to \infty$.

8. Show that if $np(n) \to 0$ as $n \to \infty$ then almost every graph in $G(n,p)$ is a forest. [Hint: Markov]

Solution: Let $X$ count the number of cycles in $G(n,p)$. For $k = 3, \ldots, n$, let $X_k$ be the number of $k$-cycles in $G(n,p)$. From a result in class, $E[X_k] \leq n^k p^k$. Thus, $E[X] = \sum E[X_k] = \sum_{k=3}^{n} (np)^{k} \leq \sum_{k=3}^{n} (np)^{k} \leq (np)^{3} / (1-np) \to 0$ since $np \to 0$ as $n \to \infty$. By Markov’s inequality, $P[X \geq 1] \leq E[X] \to 0$ and thus, almost every graph contains 0 cycles, i.e., the graph is a forest.

9. Let $k \in \mathbb{N}$ be fixed. Does the property of containing any tree on $k$ vertices have a threshold function in $G(n,p)$ (we are not considering a fixed tree, as discussed in class)? If so, which one? If not, why not?

Solution: The threshold function for the property of containing any tree of on $k$ vertices is $t(n) = (1/n)^{\frac{k}{k-1}}$. There are only a finite number of non-isomorphic trees on $k$ vertices, say, $T_1, \ldots, T_\ell$. Suppose $p(n)/t(n) \to 0$, then from the result in class, since $t(n)$ is the threshold function for each $T_i$, almost every graph does not contain a copy of $T_i$. Thus, given any $\epsilon > 0$ for each $i = 1, \ldots, \ell$, there exists $n_i \in \mathbb{N}$ such that the probability that $G \in G(n,p)$ contains $T_i$ is at most $\varepsilon/\ell$ for all $n \geq n_i$. Therefore, for $n \geq \max\{n_1, \ldots, n_\ell\}$ the probability that it contains any one of these trees is at most $\ell \cdot \varepsilon / \ell = \epsilon$. Thus, almost every graph does not contain a copy of one of $T_1, \ldots, T_\ell$ when $p(n)/t(n) \to 0$.

Suppose now $p(n)/t(n) \to \infty$; since $t(n)$ is the threshold function for each $T_1$, almost every graph contains a copy of $T_1$ and hence almost every graph contains a tree on $k$ vertices, namely $T_1$. 

10. Let $k \in \mathbb{N}$ be fixed. Determine the threshold function for containing the $k$-dimensional cube $Q_k$ in $G(n,p)$. (See Problem 10 on HW 1).

Solution: $Q_k$ is a $k$-regular graph. Thus, $\epsilon(Q_k) = k/2$. If we remove edges, then we only decrease the ratio of the edges to vertices. If we remove vertices, we decrease some of the degrees, and so the ratio of the edges to vertices decreases, because this is equal to ratio of the sum of the degrees and 2 times the number of vertices. Thus, $\epsilon'(Q_k) = k/2$. Therefore, the threshold function for containing a copy of $Q_k$ is $(1/n)^{2/k}$. 
