AMS 550.472/672: Graph Theory
Homework Problems - Week XII

Problems to be handed in on Wednesday, April 20: 2,5,8.

1. What is the probability that a random graph in $\mathcal{G}(n,p)$ has exactly m edges for $0 \leq m \leq \binom{n}{2}$.

2. Let $n \in \mathbb{N}$, $p \in [0,1]$ be fixed. Let $K \subseteq \{1,\ldots,n\}$ be a fixed subset of size k and let H be a fixed simple graph on the vertex set K with ℓ edges. What is the probability that H is a subgraph of a graph in $\mathcal{G}(n,p)$? What is the probability that H is an induced subgraph of a graph in $\mathcal{G}(n,p)$?

3. What is the expected number of edges in $G \in \mathcal{G}(n,p)$?

4. Consider a probability space where the sample space is the set of all $n!$ permutations of the set of elements $\{1,\ldots,n\}$, and each permutation has probability $\frac{1}{n!}$. We say a permutation σ fixes an element $i \in \{1,\ldots,n\}$ if $\sigma(i) = i$. Find the expected number of fixed points in this probability space (i.e., find the expectation of the random variable which counts the number of fixed points in a permutation).

5. Let $r \in \mathbb{N}$ be a fixed natural number. What is the expected number of K_r-subgraphs in $G \in \mathcal{G}(n,p)$?

6. Let $k \in \mathbb{N}$ be a fixed natural number. Compute the expected number of degree k vertices in $G \in \mathcal{G}(n,p)$.

7. Let $k \geq 2$ be a natural number. Let H be a fixed subgraph on $\{1,\ldots,k\}$, and let h be the number of graphs on $\{1,\ldots,k\}$ that are isomorphic to H. Show that $h \leq k!$. Give an example of H where this inequality is strict, but H is not the complete graph. Give an example where this inequality is an equality; can you do this with H being a tree?

8. Find the expected number of spanning trees in $\mathcal{G}(n,p)$. (Hint: Recall from the Matrix-tree theorem that the number of (labeled) spanning trees in K_n is n^{n-2})