AMS 550.472/672: Graph Theory
Homework Problems - Week XI

Problems to be handed in on Wednesday, April 13: 5(iii), 6, 7, 8. You may use any other HW problem's result without proof.

1. Prove that $\chi(T; k) = k(k-1)^{n-1}$ when T is a tree on n vertices.
2. Let C_n denote the cycle graph on n vertices. Show that $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$.
3. Do 5.3.1, 5.3.11, 5.3.18a) from the textbook.
4. Let G be a connected, simple, graph and let $k \geq 3$. Show that $\chi(G; k) \leq k(k-1)^{n-1}$. Further, show that if G is NOT a tree, then $\chi(G; k) < k(k-1)^{n-1}$. [Thus, amongst connected graphs, $\chi(G; k) = k(k-1)^{n-1}$ for $k \geq 3$ is another characterization of trees].
5. **Coefficients of $\chi(G; k)$.** Let G be a simple graph with n vertices.
   (i) Prove that the last nonzero term in the chromatic polynomial of a graph G is the term whose exponent equals the number of connected components of G.
   (ii) Conclude that the chromatic polynomial has no constant term, i.e., k is always a factor of $\chi(G; k)$.
   (iii) Let $\chi(G; k) = \sum_{i=0}^{n-1}(-1)^i a_i k^{n-i}$. Show that if G is connected then $a_i \geq \binom{n-1}{i}$ for every $0 \leq i \leq n-1$. (You may use without proof the identity $\binom{m-1}{i-1} + \binom{m-1}{i} = \binom{m}{i}$.)[Hint: use induction in a careful way, and the chromatic polynomial recurrence.] **Note: The range of i should be $0 \leq i \leq n-1$, as opposed to the earlier version of the question where is was $1 \leq i \leq n$. Thanks to Yi Wang for pointing this out.**
6. Show that if $p(k) = k^n - ak^{n-1} + \cdots \pm ck^r$ and $a > \binom{n-r+1}{2}$, then $p(k)$ cannot be a chromatic polynomial for some graph. Use this to show that $k^4 - 4k^3 + 3k^2$ is not a chromatic polynomial. [Hint: What is the maximum number of edges in a graph on n vertices with r components ? You may use the identity $\binom{a}{2} + \binom{b}{2} \leq \binom{a+b-1}{2}$ when $a, b \geq 2$, if needed]
7. Let G be a graph. Prove that some subset of vertices of the line graph forms a clique if and only if the corresponding edges all have a common end-point, or form a triangle.
8. Show that for a bipartite graph G, $\chi'(G) = \Delta(G)$.