Problems to be handed in on Wednesday, Feb 3, 2016 in class: 3, 7, 9, 11.

1. Let $G$ be a simple graph with $n$ vertices and $m$ edges. Show that $m \leq \binom{n}{2}$, and determine when equality holds.

2. What is the maximum possible degree in a SIMPLE graph on $n$ vertices? What is the minimum possible degree?

3. Show that in any simple graph, there exist two vertices with the same degree.

4. $(n$-CUBE) Define a graph $Q_n$ ($n \geq 1$), called the $n$-cube, with vertex set $V(Q_n)$ being the set of all $n$-tuples of $\{0, 1\}$, and two vertices are adjacent if and only if the corresponding tuples differ in exactly 1 coordinate.
   
   (i) Draw $Q_1, Q_2, Q_3, Q_4$.

5. The boolean lattice $BL_n$ ($n \geq 1$) is the graph whose vertex set is the set of all subsets of $\{1, \ldots, n\}$, where two subsets $X,Y$ are adjacent if and only if their symmetric difference has precisely one element.
   
   (i) Draw $BL_1, BL_2, BL_3, BL_4$.

6. Write the adjacency matrix and incidence matrix for the following graphs:
   
   (i) $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{e_1, e_2, e_3\}$, $\phi(e_1) = v_1v_2$, $\phi(e_2) = v_1v_3$, $\phi(e_3) = v_1v_4$
   
   (ii) The graph with the following drawing:

   ![Graph Diagram]

7. Show that two isomorphic simple graphs have the same number of edges. Show that any isomorphism maps a vertex to another vertex of the same degree.

8. Write the adjacency matrices of the graphs in Problem 1.1.18 in the textbook.

9. Let $G$ be a simple graph with $n$ vertices and $A$ be its $n \times n$ adjacency matrix. Determine a vector $p \in \mathbb{R}^n$ such that $q = Ap$ is the vector of the degrees, i.e., $q_i$ is the degree of vertex $i$, $i = 1, \ldots, n$.

10. Do Problem 1.1.18 from the textbook.

11. Show that for every natural number $n \in \mathbb{N}$, the $n$-CUBE $Q_n$ and the boolean lattice $BL_n$ are isomorphic.

12. Determine which pairs of the following graphs are isomorphic. Justify the answer.
13. Determine if the following adjacency matrices are of isomorphic graphs.

\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

14. Let \( G, H \) be simple graphs with adjacency matrices \( A(G) \) and \( A(H) \) respectively. Given just the adjacency matrices, can you see a way to test if \( G \) is isomorphic to \( H \)? (Think about permuting rows and columns).