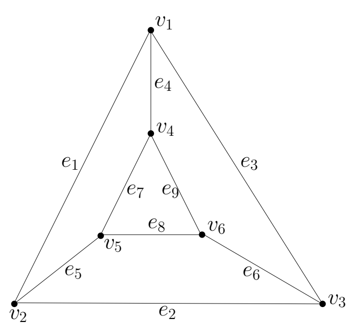
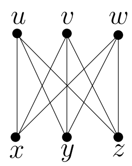
AMS 550.472/672: Graph Theory
Homework Problems - Week I

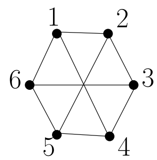
Problems to be handed in on Wednesday, Feb 3, 2016 in class: 3, 7, 9, 11.

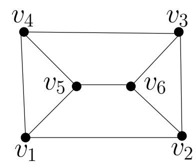
1. Let G be a simple graph with n vertices and m edges. Show that $m \leq \binom{n}{2}$, and determine when equality holds.
2. What is the maximum possible degree in a SIMPLE graph on n vertices ? What is the minimum possible degree ?
3. Show that in any simple graph, there exist two vertices with the same degree.
4. (n-CUBE) Define a graph Q_n ($n \geq 1$), called the n-cube, with vertex set $V(Q_n)$ being the set of all n-tuples of $\{0, 1\}$, and two vertices are adjacent if and only if the corresponding tuples differ in exactly 1 coordinate.
   (i) Draw Q_1, Q_2, Q_3, Q_4.
5. The *boolean lattice* BL_n ($n \geq 1$) is the graph whose vertex set is the set of all subsets of $\{1, \ldots, n\}$, where two subsets X, Y are adjacent if and only if their symmetric difference has precisely one element.
   (i) Draw BL_1, BL_2, BL_3, BL_4.
6. Write the adjacency matrix and incidence matrix for the following graphs:
   (i) $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{e_1, e_2, e_3\}$, $\phi(e_1) = v_1v_2$, $\phi(e_2) = v_1v_3$, $\phi(e_3) = v_1v_4$
   (ii) The graph with the following drawing:

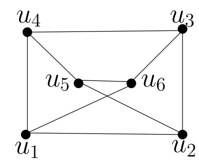


7. Show that two isomorphic *simple* graphs have the same number of edges. Show that any isomorphism maps a vertex to another vertex of the same degree.
8. Write the adjacency matrices of the graphs in Problem 1.1.18 in the textbook.
9. Let G be a simple graph with n vertices and A be its $n \times n$ adjacency matrix. Determine a vector $p \in \mathbb{R}^n$ such that $q = Ap$ is the vector of the degrees, i.e., q_i is the degree of vertex i, $i = 1, \ldots, n$.
10. Do Problem 1.1.18 from the textbook.
11. Show that for every natural number $n \in \mathbb{N}$, the n-CUBE Q_n and the boolean lattice BL_n are isomorphic.
12. Determine which pairs of the following graphs are isomorphic. Justify the answer.









13. Determine if the following adjacency matrices are of isomorphic graphs.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

14. Let G, H be simple graphs with adjacency matrices $A(G)$ and $A(H)$ respectively. Given just the adjacency matrices, can you see a way to test if G is isomorphic to H ? (Think about permuting rows and columns).