Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Spring Session

Tuesday, January 18, 2011

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Show that the function $f(x) = x^{-1}$ is *not* uniformly continuous for $0 < x < 1$.

Solution: For $0 < x < x + \delta < 1$ we have

$$|f(x) - f(x+\delta)| = \left| \frac{1}{x} - \frac{1}{x+\delta} \right| = \frac{\delta}{x(x+\delta)}.$$

Given $\epsilon > 0$, there is no δ for which this quantity can be bounded by ϵ independently of x.

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2. Fix integer $n \geq 0$. Let X be a random variable whose distribution is Binomial(n, p) under probability measure $\mathbb{P}_p$ (with corresponding expectation operator $\mathbb{E}_p$), for each $p \in (0, 1)$. Prove that there does not exist a function $g : \{0, \ldots, n\} \to \mathbb{R}$ such that $\mathbb{E}_p[g(X)] = \frac{p}{1-p}$ for all $p \in (0, 1)$.

Solution: The left-hand side $\mathbb{E}_p[g(X)] = \sum_{x=0}^{n} g(x) \binom{n}{x} p^x (1-p)^{n-x}$ is a polynomial in p, while the right-hand side $\frac{p}{1-p}$ is not.

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3. Let

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

(a) Determine whether A is diagonalizable.

(b) Determine whether A is orthogonally diagonalizable.

Solution:

(a) Since A is upper triangular, we see that its eigenvalues are $-2, -2, 3, 3$. A quick calculation yields $(A + 2I)x = 0$ for all $x \in \text{span}\{[1, 0, 0, 0]^T, [0, 1, 0, 0]^T\}$, $(A - 3I)x = 0$ for all $x \in \text{span}\{[0, 1, 1, 0]^T, [0, 0, 1, 1]^T\}$. Thus A has 4 linearly independent eigenvectors and is diagonalizable.

(b) If A is orthogonally diagonalizable, then A is normal: $AA^T = A^T A$. Since

$$AA^T = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 54 & 15 & -15 \\ 0 & 15 & 9 & 0 \\ 0 & -15 & 0 & 9 \end{bmatrix} \neq A^T A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & -10 & 10 \\ 0 & -10 & 34 & -25 \\ 0 & 10 & -25 & 34 \end{bmatrix},$$

A is not orthogonally diagonalizable.

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4. Suppose that $U \sim \text{Gamma}(\alpha, \lambda)$ and $V \sim \text{Gamma}(\beta, \lambda)$ are independent Gamma random variables. Show that

$$X := \frac{U}{U+V}, \quad Y := \frac{1}{U+V}$$

are also independent random variables and determine their densities.

Solution: It is easy to invert the above definitions to give

$$U = \frac{X}{Y}, \quad V = \frac{1-X}{Y}.$$

The Jacobian determinant of the transformation is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1/y & -x/y^2 \\ -1/y & -(1-x)/y^2 \end{vmatrix} = \begin{vmatrix} 1/y & 0 \\ -1/y & -1/y^2 \end{vmatrix} = -\frac{1}{y^3},$$

where we added x/y times the first column to the second column. Thus, for $0 < x < 1$ and $y > 0$, the change-of-variables formula gives for the joint density

$$\begin{aligned} p_{X,Y}(x,y) &= p_U(\tfrac{x}{y})\, p_V(\tfrac{1-x}{y})\, \tfrac{1}{y^3} \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{x}{y}\right)^{\alpha-1} e^{-\lambda x/y} \cdot \frac{\lambda^\beta}{\Gamma(\beta)} \left(\frac{1-x}{y}\right)^{\beta-1} e^{-\lambda(1-x)/y} \cdot \frac{1}{y^3} \\ &= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} y^{-(\alpha+\beta+1)} e^{-\lambda/y} \\ &= p_X(x)\, p_Y(y) \end{aligned}$$

with

$$p_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

and

$$p_Y(y) = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)} y^{-(\alpha+\beta+1)} e^{-\lambda/y}.$$

The normalizations can be obtained either by using the definition of the gamma function and the change of variables $z = 1/y$ in the density for Y (which is, in fact, an Inverse Gamma density) or by using the definition of the beta function in the density for X (which is, in fact, a Beta density). It follows that X and Y are independent random variables with the above densities.

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5. Let A and B be real symmetric $n \times n$ matrices. Suppose that λ is an eigenvalue of $AB - BA$. Prove that the real part of λ is zero.

Solution: Let λ be an eigenvalue corresponding to the (nonzero) eigenvector v of $AB - BA$. Then

$$v^*(AB - BA)v = v^*\lambda v = \lambda\|v\|^2.$$

Taking conjugate transposes we find

$$\left[v^*(AB - BA)v\right]^* = \overline{\lambda}\|v\|^2;$$

but also

$$\begin{aligned}\left[v^*(AB - BA)v\right]^* &= v^*(AB - BA)^*v \\ &= v^*(B^*A^* - A^*B^*)v \\ &= v^*(BA - AB)v \\ &= -\lambda\|v\|^2.\end{aligned}$$

Therefore $\overline{\lambda} = -\lambda$. So if $\lambda = a + b\,\mathrm{i}$ then we have $a - b\,\mathrm{i} = -(a + b\,\mathrm{i})$, yielding $a = 0$.

Simplified Solution: With notations as above,

$$\lambda\|v\|^2 = \langle v, (AB - BA)v\rangle = \langle Av, Bv\rangle - \langle Bv, Av\rangle = 2i\,\mathrm{Im}\langle Av, Bv\rangle,$$

because real, symmetric matrices are hermitian. Thus, λ is pure imaginary.

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6. For each $x > 0$ and each $n = 1, 2, \ldots$, show that

$$x^{n+1} + \frac{1}{x^{n+1}} \geq x^n + \frac{1}{x^n}.$$

Solution: Rearranging terms, the inequality is equivalent to

$$x^{2n+1}(x - 1) \geq (x - 1)$$

which can be verified separately for the three cases (i) $0 < x < 1$, (ii) $x = 1$, and (iii) $x > 1$.

7. Let $\mathbb{P}(A)$ and $\mathbb{P}(B)$ denote the probabilities of events A and B. Prove that

$$|\mathbb{P}(A) - \mathbb{P}(B)| \leq \max\{\mathbb{P}(A \setminus B), \mathbb{P}(B \setminus A)\}.$$

NOTE: Here $X \setminus Y$ is used (without assuming that $Y \subseteq X$) to denote set difference.

Solution:

(*Solution #1*:) Note that since sets A and B are events (i.e., measurable sets), the same is true of the sets $A \setminus B$, $B \setminus A$, and $A \cap B$. From the disjoint-union representations

$$A = (A \setminus B) \cup (A \cap B) \qquad \text{and} \qquad B = (B \setminus A) \cup (A \cap B),$$

it follows that

$$\mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) \qquad \text{and} \qquad \mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B),$$

so that

$$\begin{aligned} |\mathbb{P}(A) - \mathbb{P}(B)| &= |\mathbb{P}(A \setminus B) - \mathbb{P}(B \setminus A)| \\ &= \max\{\mathbb{P}(A \setminus B),\ \mathbb{P}(B \setminus A)\} - \min\{\mathbb{P}(A \setminus B),\ \mathbb{P}(B \setminus A)\} \\ &\leq \max\{\mathbb{P}(A \setminus B),\ \mathbb{P}(B \setminus A)\}, \end{aligned}$$

as desired.

NOTE: When will *equality* hold? The argument shows that this occurs if and only if $\min\{\mathbb{P}(A \setminus B), \mathbb{P}(B \setminus A)\} = 0$, i.e., if and only if one of the events A and B is almost surely contained in the other.

(*Solution #2*:) Note that since sets A and B are events (i.e., measurable sets), the same is true of the sets $A \setminus B$ and $B \setminus A$. From the inclusion

$$A \subseteq B \cup (A \setminus B)$$

it follows by finite subadditivity of $\mathbb{P}$ that

$$\mathbb{P}(A) \leq \mathbb{P}(B) + \mathbb{P}(A \setminus B);$$

reversing the roles of A and B we also have

$$\mathbb{P}(B) \leq \mathbb{P}(A) + \mathbb{P}(B \setminus A).$$

Thus

$$|\mathbb{P}(A) - \mathbb{P}(B)| = \max\{\mathbb{P}(A) - \mathbb{P}(B),\ \mathbb{P}(B) - \mathbb{P}(A)\} \leq \max\{\mathbb{P}(A \setminus B),\ \mathbb{P}(B \setminus A)\},$$

as desired.

8. For integer $n \geq 0$, show that

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)}$$

for $0 < \theta < 2\pi$.

Solution:

(*Solution #1*:) We have

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \mathrm{Re}\left\{\sum_{k=0}^{n}(e^{i\theta})^k\right\} = \mathrm{Re}\left\{\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right\}.$$

Combining this with

$$\mathrm{Re}\left\{\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right\} = \mathrm{Re}\left\{\frac{e^{-i\theta/2}-e^{i[n+(1/2)]\theta}}{e^{-i\theta/2}-e^{i\theta/2}}\right\} = \frac{\sin(\theta/2)+\sin[(n+\frac{1}{2})\theta]}{2\sin(\theta/2)},$$

we obtain the result.

(*Solution #2*:) Prove the formula by induction on n. For $n = 0$, it reduces to $1 = \frac{1}{2}+\frac{1}{2}$. The induction step (from $n-1$ to n) follows from

$$\sin[(n+\tfrac{1}{2})\theta] - \sin[(n-\tfrac{1}{2})\theta] = 2\cos(n\theta)\sin(\tfrac{\theta}{2}),$$

which is an instance of the trigonometric identity $\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$.

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9. Two types of radios are available to be carried on a hiking expedition, but only one radio can be taken. The expedition wants to choose the radio that has the longest expected operating time on one set of batteries. Radio A takes a set of 2 batteries, each of which has an expected lifetime of 200 hours, and operates if at least one of its batteries operates. Radio B uses a set of 4 batteries, but the expected lifetime of each battery is 400 hours, and the radio operates only if at least 3 of the batteries operate. Which radio should the expedition take? Justify your answer carefully. Assume that the lengths of useful life of the batteries follow Exponential probability distributions and are independent of each other.

Solution: For radio A, the time until one of the batteries fails is the minimum of two Exponential(1/200) random variables. The minimum of independent Exponential random variables has an Exponential distribution with the sum of the parameter values as its parameter. Thus the time until first failure of a battery is an Exponential(1/100) random variable. By the lack of memory property, the additional time until failure of the second battery is still Exponential(1/200). The expected value of the sum of the two "interfailure" random variables is $100 + 200 = 300$.

For radio B, by similar reasoning, the time until one of the batteries fails is an Exponential(1/100) random variable, as the minimum of four Exponential(1/400) random variables. By the lack of memory property, the remaining lifetimes of the other three batteries are independent Exponential(1/400) random variables. The additional time until a second battery fails is the minimum of these three random variables, which is distributed Exponential(3/400). The expected value of the sum is $100 + 133\frac{1}{3} = 233\frac{1}{3} < 300$.

Therefore, the expedition should choose radio A.

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10. Suppose that $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^n$, and consider the set S of points $\mathbf{x} \in \mathbb{R}^n$ characterized by having distance from $\mathbf{a}$ which is twice the distance from $\mathbf{b}$. Show that S is a sphere of radius $r \geq 0$ centered at a point $\mathbf{c} \in \mathbb{R}^n$, and find r and $\mathbf{c}$.

Solution: In symbols, $\mathbf{x} \in S$ if and only if $|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$. Squaring gives the equivalent condition $|\mathbf{x} - \mathbf{a}|^2 = 4|\mathbf{x} - \mathbf{b}|^2$, or, gathering like terms,

$$3|\mathbf{x}|^2 - 2(4\mathbf{b} - \mathbf{a}) \cdot \mathbf{x} + 4|\mathbf{b}|^2 - |\mathbf{a}|^2 = 0.$$

Dividing by 3 and completing the square gives

$$\left|\mathbf{x} - \left(\tfrac{4}{3}\mathbf{b} - \tfrac{1}{3}\mathbf{a}\right)\right|^2 = \tfrac{4}{9}|\mathbf{a} - \mathbf{b}|^2.$$

Thus,

$$\mathbf{c} = \tfrac{4}{3}\mathbf{b} - \tfrac{1}{3}\mathbf{a}, \quad r = \tfrac{2}{3}|\mathbf{a} - \mathbf{b}|.$$

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11. Let u, v be two unit vectors in $\mathbb{R}^n$, and let $a, b \in \mathbb{R}$. Compute

$$\det(I + auu^T + bvv^T),$$

where I denotes the n-by-n identity matrix.

Solution: Let $A := I + auu^T + bvv^T$. We will express A in a suitable basis of $\mathbb{R}^n$.

First assume that u and v are linear dependent, which implies $v = \pm u$ since both vectors have unit norm, and hence $A = I + (a + b)uu^T$. If one extends $\{u\}$ to an orthonormal basis, the matrix of (the linear operator associated to) A in this basis is diagonal, with first diagonal coefficient equal to $1 + a + b$ and the rest equal to 1. So the determinant is $1 + a + b$ in this case.

In the general-position case, complete $\{u, v\}$ into a basis $\{u, v, x_3, \ldots, x_n\}$ where $\{x_3, \ldots, x_n\}$ are orthogonal to the space spanned by $\{u, v\}$. We have the equations $Au = (1 + a)u + b(v^T u)v$, $Av = a(v^T u)u + (1 + b)v$ and $Ax_i = x_i$ for $i = 3, \ldots, n$. This implies that

$$\det A = \det \begin{pmatrix} 1 + a & a(v^T u) \\ b(v^T u) & 1 + b \end{pmatrix} = (1 + a)(1 + b) - ab(u^T v)^2.$$

Observe that the equation $\det A = (1 + a)(1 + b) - ab(u^T v)^2$ gives the correct answer in either case.

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12. Let $b > a > 0$.

(a) Let $y > 0$. By considering the integral

$$\int_0^y \int_a^b e^{-tx}\,dt\,dx,$$

show that

$$\int_0^y \frac{e^{-ax} - e^{-bx}}{x}\,dx = \int_a^b \frac{1 - e^{-ty}}{t}\,dt.$$

(b) Use the result of part (a) to calculate

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx.$$

Solution:

(a) Part (a) is immediate using a change of order of integration.

(b) The inequality $\int_a^b \frac{e^{-ty}}{t}\,dt \le e^{-ay}\log(\frac{b}{a})$ follows from $0 \le e^{-ty}/t \le e^{-ay}/t$ for $t \in [a, b]$. Taking $y \to \infty$, we see that

$$\int_0^y \frac{e^{-ax} - e^{-bx}}{x}\,dx = \int_a^b \frac{1}{t}\,dt - \int_a^b \frac{e^{-ty}}{t}\,dt \to \log\left(\frac{b}{a}\right)$$

as $y \to \infty$.

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13. Let $z = [z_1, \ldots, z_n]^T$ be a vector of real numbers and let A be an $n \times n$ real orthogonal matrix whose first row is $[\,1/\sqrt{n}, \ldots, 1/\sqrt{n}\,]$. Define $y = [y_1, \ldots, y_n]^T$ via $y := Az$. Let $\overline{z} := (1/n)\sum_{k=1}^n z_k$. Show that

$$\sum_{k=1}^n (z_k - \overline{z})^2 = \sum_{k=2}^n y_k^2.$$

Solution: From the definition of A we see that

$$y_1 = (1/\sqrt{n})\sum_{k=1}^n z_k = \sqrt{n}\,\overline{z}.$$

Also,

$$\sum_{k=1}^{n} y_k^2 = \|y\|_2^2 = \|Az\|_2^2 = \|z\|_2^2 \text{ (because } A \text{ is orthogonal)} = \sum_{k=1}^{n} z_k^2.$$

Thus

$$\sum_{k=1}^{n}(z_k - \overline{z})^2 = \sum_{k=1}^{n} z_k^2 - n\overline{z}^2 = \sum_{k=1}^{n} z_k^2 - y_1^2 = \sum_{k=1}^{n} y_k^2 - y_1^2 = \sum_{k=2}^{n} y_k^2.$$

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14. Suppose that Babyface challenges Scarface to a simple gambling game, in which they roll fair six-sided dice independently. Babyface will roll the die 10 times, but Scarface gets to roll the die 11 times. If Scarface rolls (strictly) more 6's than Babyface, Babyface will pay Scarface \$100. Otherwise, Scarface will pay Babyface \$100. Should Scarface accept the challenge?

Solution: Condition on the results after each has rolled the die 10 times. Let B denote the number of 6's rolled by Babyface and S denote the number of 6's rolled by Scarface.

Clearly,

$$\mathbb{P}[B > S] + \mathbb{P}[B = S] + \mathbb{P}[B < S] = 1.$$

However, by symmetry, $\mathbb{P}[B > S] = \mathbb{P}[B < S]$.

On the event $\{B > S\}$, Babyface wins, since the best Scarface can do is tie on the number of 6's.

On the event $\{B < S\}$, Scarface wins, even without making the last roll.

On the event $\{B = S\}$, Scarface wins if he gets a 6 on his last roll, which happens with probability $1/6$, and loses otherwise.

Thus,

$$\mathbb{P}[\text{Scarface wins}] = \mathbb{P}[B < S] + \frac{1}{6}\mathbb{P}[B = S]$$

and

$$\mathbb{P}[\text{Babyface wins}] = \mathbb{P}[B > S] + \frac{5}{6}\mathbb{P}[B = S],$$

so

$$\mathbb{P}[\text{Babyface wins}] > \mathbb{P}[\text{Scarface wins}].$$

Therefore, Scarface should not accept the challenge.

15. Assume that A and B are m-by-n and n-by-m real matrices, respectively. Prove that $\det(I_m - AB) = \det(I_n - BA)$, where I_r denotes the r-by-r identity matrix.

Solution:

(*Solution #1*:) Write

$$\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \begin{pmatrix} I_m & 0 \\ -B & I_n \end{pmatrix} = \begin{pmatrix} I_m - AB & A \\ 0 & I_n \end{pmatrix}$$

and

$$\begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \begin{pmatrix} I_m & -A \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ B & I_n - BA \end{pmatrix}$$

to reach the desired conclusion:

$$\begin{aligned}
\det(I_m - AB) &= \det(I_m - AB) \times \det I_n = \det \begin{pmatrix} I_m - AB & A \\ 0 & I_n \end{pmatrix} \\
&= \det \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \times \det \begin{pmatrix} I_m & 0 \\ -B & I_n \end{pmatrix} \\
&= \det \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \times \det \begin{pmatrix} I_m & -A \\ 0 & I_n \end{pmatrix} \\
&= \det \begin{pmatrix} I_m & 0 \\ B & I_n - BA \end{pmatrix} = \det I_m \times \det(I_n - BA) \\
&= \det(I_n - BA).
\end{aligned}$$

(*Solution #2*:) Provided that the spectral radii of AB and BA are both < 1 we have

$$\begin{aligned}
\ln\det(I_m - AB) &= \operatorname{tr}\ln(I_m - AB) \\
&= -\sum_{k=1}^{\infty} \tfrac{1}{k}\operatorname{tr}[(AB)^k] \\
&= -\sum_{k=1}^{\infty} \tfrac{1}{k}\operatorname{tr}[(BA)^k] \quad \text{by cyclicity of trace} \\
&= \operatorname{tr}\ln(I_n - BA) = \ln\det(I_n - BA).
\end{aligned}$$

Since the two determinants in question are finite-degree polynomials in the entries of A and B, equality under the spectral-radii condition implies equality for all A and B.