Department of Applied Mathematics and Statistics
The Johns Hopkins University

# Introductory Examination–Spring Session

Tuesday, January 18, 2011

**Instructions: Read carefully!**

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e., 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. Show that the function $f(x) = x^{-1}$ is *not* uniformly continuous for $0 < x < 1$.

2. Fix integer $n \geq 0$. Let $X$ be a random variable whose distribution is Binomial$(n, p)$ under probability measure $\mathbb{P}_p$ (with corresponding expectation operator $\mathbb{E}_p$), for each $p \in (0, 1)$. Prove that there does not exist a function $g : \{0, \dots, n\} \to \mathbb{R}$ such that $\mathbb{E}_p[g(X)] = \frac{p}{1-p}$ for all $p \in (0, 1)$.

3. Let
$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$
   (a) Determine whether $A$ is diagonalizable.
   (b) Determine whether $A$ is orthogonally diagonalizable.

4. Suppose that $U \sim$ Gamma$(\alpha, \lambda)$ and $V \sim$ Gamma$(\beta, \lambda)$ are independent Gamma random variables. Show that
$$X := \frac{U}{U+V}, \quad Y := \frac{1}{U+V}$$
are also independent random variables and determine their densities.

5. Let $A$ and $B$ be real symmetric $n \times n$ matrices. Suppose that $\lambda$ is an eigenvalue of $AB - BA$. Prove that the real part of $\lambda$ is zero.

6. For each $x > 0$ and each $n = 1, 2, \dots$, show that
$$x^{n+1} + \frac{1}{x^{n+1}} \geq x^n + \frac{1}{x^n}.$$

7. Let $\mathbb{P}(A)$ and $\mathbb{P}(B)$ denote the probabilities of events $A$ and $B$. Prove that

$$|\mathbb{P}(A) - \mathbb{P}(B)| \leq \max\{\mathbb{P}(A \setminus B), \mathbb{P}(B \setminus A)\}.$$

NOTE: Here $X \setminus Y$ is used (without assuming that $Y \subseteq X$) to denote set difference.

8. For integer $n \geq 0$, show that

$$1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin[(n + \frac{1}{2})\theta]}{2\sin(\theta/2)}$$

for $0 < \theta < 2\pi$.

9. Two types of radios are available to be carried on a hiking expedition, but only one radio can be taken. The expedition wants to choose the radio that has the longest expected operating time on one set of batteries. Radio A takes a set of 2 batteries, each of which has an expected lifetime of 200 hours, and operates if at least one of its batteries operates. Radio B uses a set of 4 batteries, but the expected lifetime of each battery is 400 hours, and the radio operates only if at least 3 of the batteries operate. Which radio should the expedition take? Justify your answer carefully. Assume that the lengths of useful life of the batteries follow Exponential probability distributions and are independent of each other.

10. Suppose that $\mathbf{a} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^n$, and consider the set $S$ of points $\mathbf{x} \in \mathbb{R}^n$ characterized by having distance from $\mathbf{a}$ which is twice the distance from $\mathbf{b}$. Show that $S$ is a sphere of radius $r \geq 0$ centered at a point $\mathbf{c} \in \mathbb{R}^n$, and find $r$ and $\mathbf{c}$.

11. Let $u, v$ be two unit vectors in $\mathbb{R}^n$, and let $a, b \in \mathbb{R}$. Compute

$$\det(I + auu^T + bvv^T),$$

where $I$ denotes the $n$-by-$n$ identity matrix.

12. Let $b > a > 0$.

    (a) Let $y > 0$. By considering the integral

    $$\int_0^y \int_a^b e^{-tx}\,dt\,dx,$$

    show that

    $$\int_0^y \frac{e^{-ax} - e^{-bx}}{x}\,dx = \int_a^b \frac{1 - e^{-ty}}{t}\,dt.$$

    (b) Use the result of part (a) to calculate

    $$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx.$$

13. Let $z = [z_1, \ldots, z_n]^T$ be a vector of real numbers and let $A$ be an $n \times n$ real orthogonal matrix whose first row is $[\,1/\sqrt{n}, \ldots, 1/\sqrt{n}\,]$. Define $y = [y_1, \ldots, y_n]^T$ via $y := Az$. Let $\overline{z} := (1/n)\sum_{k=1}^n z_k$. Show that

    $$\sum_{k=1}^n (z_k - \overline{z})^2 = \sum_{k=2}^n y_k^2.$$

14. Suppose that Babyface challenges Scarface to a simple gambling game, in which they roll fair six-sided dice independently. Babyface will roll the die 10 times, but Scarface gets to roll the die 11 times. If Scarface rolls (strictly) more 6's than Babyface, Babyface will pay Scarface \$100. Otherwise, Scarface will pay Babyface \$100. Should Scarface accept the challenge?

15. Assume that $A$ and $B$ are $m$-by-$n$ and $n$-by-$m$ real matrices, respectively. Prove that $\det(I_m - AB) = \det(I_n - BA)$, where $I_r$ denotes the $r$-by-$r$ identity matrix.