Department of Applied Mathematics and Statistics
The Johns Hopkins University

INTRODUCTORY EXAMINATION–SPRING SESSION

Tuesday, January 12, 2010

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e. 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability;). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. A complex number $z \in \mathbb{C}$ is said to be *algebraic* if there are $k+1$ integers $n_0, ..., n_k$ with $n_k \neq 0$ such that

$$n_k z^k + \cdots + n_1 z + n_0 = 0.$$

Prove that the set of algebraic numbers in the real interval $[0,1]$ forms a dense subset of $[0,1]$ but that there are uncountably many non-algebraic numbers in $[0,1]$.

Solution: (i) Rational numbers are dense in $[0,1]$ but $\mathbb{Q} \subset \mathbb{A}$ because $r = p/q$ for integer p, q satisfies $qr - p = 0$.

(ii) It is enough to show that $\mathbb{A}$ is countable, since $[0,1]$ is uncountable. For each positive integer N, there are only a finite number of integers $n_0, ..., n_k$ and $k \geq 1$, such that

$$k + |n_0| + \cdots |n_k| = N. \tag{$*$}$$

For any choice with $n_k \neq 0$, the polynomial $P(z) = n_k z^k + \cdots + n_1 z + n_0$ has k complex roots (including multiple roots) by the Fundamental Theorem of Algebra. Thus, for any positive integer N, there is a finite set of algebraic numbers $z_1, ..., z_{m_N}$ arising as roots of polynomials whose coefficients satisfy (*). One may thus enumerate $\mathbb{A}$ by first listing the distinct solutions for $N = 1$, then the new solutions for $N = 2$, and then for $N = 3, 4, 5, ...$

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2. Fix an arbitrary $r \in \mathbb{N}$. Let p_n denote the probability when a fair coin is tossed (independently) n times that a run of r consecutive heads never appears. Observe that $p_0 = p_1 = \cdots = p_{r-1} = 1$. Prove that

$$p_n = \sum_{k=1}^{r} 2^{-k} p_{n-k}, \quad n \geq r.$$

Solution: This is an extension of a Theoretical Exercise in Chapter 2 of Ross. Let B_n denote the event that a run of r consecutive heads never appears in the first n tosses of an infinite sequence, and let A_k denote the event that this sequence of tosses begins with a run of $k-1$ consecutive heads followed by a tails. Then $P(A_k) = 2^{-k}$ and for $n \geq r$ we have

$$B_n = \cup_{k=1}^{r} (A_k \cap B_n),$$

which is a union of disjoint events. Using this and independence gives

$$p_n = P(B_n) = \sum_{k=1}^{r} P(A_k \cap B_n) = \sum_{k=1}^{r} P(A_k) P(B_n | A_k) = \sum_{k=1}^{r} 2^{-k} P(B_{n-k}) = \sum_{k=1}^{r} 2^{-k} p_{n-k},$$

as desired.

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3. Let $g(x) > 0$ be a function on $[0, \infty)$ for which

$$\int_0^\infty xg(x)dx < \infty$$

and define

$$a_n = \int_n^\infty g(x)dx, \quad n = 1, 2, ...$$

Determine the convergence or divergence of $\sum_n a_n$.

Solution:

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} \int_n^\infty g(x)dx \\
&= \int_1^\infty \left(\sum_{n=1}^{[x]} 1 \right) g(x)dx \\
&\leq \int_1^\infty [x]g(x)dx \\
&< \infty
\end{aligned}$$

where $[x]$ is the greatest integer less than or equal to x.

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4. Let S and T be two subsets in $\mathbb{R}^n$ defined as follows:

$$S = \{\ x\ : x_1 = x_2 = \cdots = x_n\}; \quad T = \{\ x\ : x_1 + x_2 + \cdots + x_n = 0\}.$$

(a) Show that S and T are subspaces.
(b) Show that for any vector $z \in \mathbb{R}^n$ there exist unique pair (x, y) such that $x \in S$ and $y \in T$ and $z = x + y$.

Solution:

(a) For S to be a subspace, just check (i) $x, y \in S \Longrightarrow x + y \in S; (ii) x \in S,\ \alpha \in R \Longrightarrow \alpha x \in S$. Do the same check for T.

(b) Define $\alpha = \frac{1}{n}(z_1 + \cdots + z_n)$, $x = \alpha e$ and $y = z - x$ where $e^T = (1, 1, \cdots, 1)$. Clearly, $x \in S$ and $z = x + y$. We have $e^T y = e^T z - \alpha e^T e = z_1 + \cdots + z_n - n\alpha = 0$. Hence $y \in T$. To show the uniqueness, we first observe that $S \cap T = \{0\}$. If we also have $\hat{x} \in S, \hat{y} \in T$ and $z = \hat{x} + \hat{y} = x + y$. Then, $\hat{x} - x = y - \hat{y}$. The left-hand-side is a vector in S and the right-hand-side is a vector in T. Therefore, $\hat{x} - x = y - \hat{y} \in S \cap T$ and is zero. This proves the uniqueness of the decomposition.

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5. Let $y \neq 0$ in $\mathbb{R}^n$, and define $A = yy^T$, an $n \times n$ matrix. Find all the eigenvalues of A and find the algebraic and geometric multiplicities of each eigenvalue.

Solution: Since $Ay = (\|y\|_2^2)y$, $\lambda_1 = \|y\|_2^2$ is an eigenvalue of A and its geometric multiplicity is at least 1.

Since for any $x \perp y$, i.e. $< x, y >= 0$, we have $Ax = yy^T x = (< x, y >)y = 0$, $\lambda_2 = 0$ is an eigenvalue of A and its geometric multiplicity is at least $n - 1$.

Since the algebraic multiplicity of an eigenvalue is bounded below by its geometric multiplicity and since the sum of algebraic multiplicities of the eigenvalues of an $n \times n$ matrix is always n, we see that the eigenvalues λ_1 and λ_2 we found above are the only eigenvalues of A, and the algebraic and geometric multiplicity of λ_1 is 1, and of λ_2 is $n - 1$.

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6. Prove the following statement or give a counterexample to show it is false:

Given an open interval $(a, b) \subset [0, +\infty)$ and a positive integer N there exists $C > 0$ such that for every $x > C$ we have $x \in (na, nb)$ for some integer $n \geq N$.

Solution: The statement is true. Take $C = \max\{\frac{1}{1/a - 1/b}, Nb\}$. Clearly this is positive. If $x > C$ we have $x > \frac{1}{1/a - 1/b}$ which implies $x/a - x/b > 1$. Since the length of the open interval $(x/b, x/a)$ exceeds 1 it must contain some positive integer n. This gives $x/b < n < x/a$ hence $x \in (na, nb)$. Furthermore, for such an x and n we have

$$n > x/b > C/b \geq Nb/b = N.$$

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7. Show that for every $n \times n$ complex matrix A which is not proportional to the identity matrix there exists at least one n-vector x, $x \neq 0$,which is not an eigenvector.

Solution: Assume that for all vectors x, there is a $\lambda_x \in \mathbb{C}$ such that $Ax = \lambda_x x$. For any two linearly independent vectors x_1, x_2, we see that $Ax_i = \lambda_{x_i} x_i,\ \ i = 1, 2$ together with

$$A(x_1 + x_2) = \lambda_{x_1+x_2}(x_1 + x_2)$$

implies that

$$(\lambda_{x_1} - \lambda_{x_1+x_2})x_1 = (\lambda_{x_1+x_2} - \lambda_{x_2})x_2.$$

By linear independence, this can only hold if $\lambda_{x_1} - \lambda_{x_1+x_2} = \lambda_{x_1+x_2} - \lambda_{x_2} = 0$, so that $\lambda_{x_1} = \lambda_{x_1+x_2} = \lambda_{x_2}$. If we take a linearly independent set $\{x_1, ..., x_n\}$ that spans the space, then we see from this that $Ax_i = \lambda x_i$ for some fixed $\lambda \in \mathbb{C}$ for all $i = 1, ..., n$. Hence, $A = \lambda I$.

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8. Let f and g be continuous functions defined on the set of real numbers, i.e., $f, g : \mathbb{R} \to \mathbb{R}$. Suppose that f and g have the same zeros and their common set of zeros is finite. That is, the sets

$$\{x \in \mathbb{R}\ :\ f(x) = 0\} \qquad \text{and} \qquad \{x \in \mathbb{R}\ :\ g(x) = 0\}$$

are equal and finite.

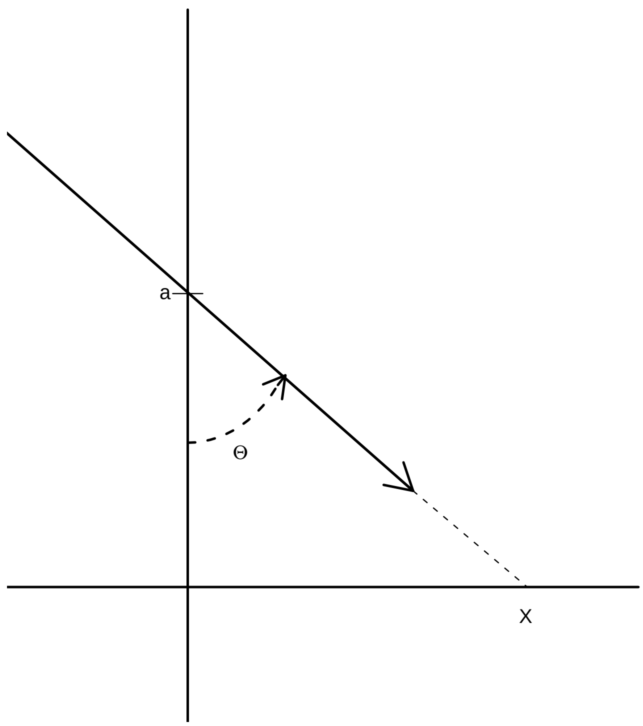
Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(a) = \lim_{x \to a} \frac{f(x)}{g(x)}.$$

Prove or disprove: h is continuous.

Solution:

h is not necessarily continuous. Take $f(x) = x$ and $g(x) = x^2$. Then $f(x)/g(x) = x/x^2 = 1/x$ for $x \neq 0$ and $\lim_{x\to 0} f(x)/g(x)$ does not exist.



9. An arrow whose center is located $a > 0$ units on the vertical axis is spun so that the angle it makes with the vertical axis is Θ, where Θ is a random variable uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (see figure). Determine the distribution of the x-coordinate that the arrow points to.

Solution: We are told Θ has the p.d.f.

$$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi} & \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases},$$

and we are asked to compute the distribution of $X = a\tan(\Theta)$. This transformation is one-to-one as θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and has inverse $\theta = \tan^{-1}(\frac{x}{a})$. Furthermore, $\theta' = \frac{a}{a^2+x^2}$. Therefore,

$$f_X(x) = \frac{1}{\pi} \cdot \frac{a}{a^2+x^2} \quad \text{for } -\infty < x < \infty,$$

that is, X has a Cauchy distribution.

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10. Suppose that X and Y are independent Geometric random variables with parameter p. Find the conditional distribution of X given $X+Y=n$, where $n \geq 1$.

Solution: This problem requires familiarity with independence, conditional probability, and the geometric and negative binomial distributions.

For $k = 1, 2, 3, \ldots, n-1$,

$$P[X=k|X+Y=n] = \frac{P[X=k, X+Y=n]}{P[X+Y=n]} = \frac{P[X=k, Y=n-k]}{P[X+Y=n]},$$

by the definition of conditional probability and elementary algebra.

By independence, this

$$= \frac{P[X=k]P[Y=n-k]}{P[X+Y=n]},$$

which, substituting geometric and negative binomial probabilities, gives

$$= \frac{(pq^{k-1})(pq^{n-k-1})}{(n-1)p^2q^{n-2}},$$

where $q = 1-p$.

Note that the negative binomial probability, if the form has not been memorized, can be easily derived from basic principles.

After considerable cancellation, the final expression is

$$= \frac{1}{n-1},$$

so the conditional distribution is a discrete uniform distribution.

11. An n by n matrix P is called a projection if $P^2 = P$. Prove that if P, Q and $P + Q$ are all projections, then $PQ = 0$.

Solution: From the assumption, we get $0 = (P + Q)^2 - P - Q = PQ + QP$. Multiplying $PQ + QP = 0$ by P on the left and on the right, we get $2PQP = 0$. Finally, we can write

$$PQ = PQ(I - P) + PQP = -QP(I - P) + 0 = 0$$

since $P(I - P) = P - P^2 = 0$.

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12. If A and B are $n \times n$ real symmetric matrices, write $A \leq B$ if and only if $B - A$ is nonnegative definite.
Show that if A and B are $n \times n$ real symmetric positive definite matrices with $A \leq B$, then $B^{-1} \leq A^{-1}$.

Solution: First, A symmetric positive definite implies $x^T A^{-1} x = \max\{y : 2x^T y - y^T Ay\}$, because $y = A^{-1}x$ maximizes the quadratic form. Next, $A \leq B$ implies $2x^T y - y^T Ay \geq 2x^T y - y^T By$ for all x, y. Thus $x^T A^{-1} x \geq x^T B^{-1} x$, which implies $B^{-1} \leq A^{-1}$, as desired.

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13. The function $f : (0, 1) \to \mathbb{R}$ is continuous on the open interval $(0, 1)$, and satisfies $0 < f(x) < x$. Let a sequence of functions $f_n : (0, 1) \to \mathbb{R}$ be defined by

$$f_1(x) = f(x), \quad f_n(x) = f(f_{n-1}(x)), \quad \text{for } n \geq 2.$$

Prove that $f_n(x) \to 0$ as $n \to \infty$ for all $x \in (0, 1)$.

Solution: For any fixed $x \in (0, 1)$, define the sequence (x_n) by $x_n := f_n(x)$. This is a decreasing sequence bounded below, so it tends to a limit, say, y. Note that since $x > x_n > 0$, we have $1 > y \geq 0$. If $y > 0$, then by continuity $x_{n+1} = f(x_n) \to f(y)$, which coincides the limit of x_n. This implies $f(y) = y$, forcing $y = 0$ (Since $f(z) < z$ for any $z \in (0, 1)$).

14. Suppose $U_1, \ldots, U_n$ are iid and distributed as Uniform$(0,1)$. Find the probability density function of the limiting distribution of the random variable $\left(\prod_{i=1}^n eU_i\right)^{1/\sqrt{n}}$ as $n \to \infty$.

Solution: Let Y be the random variable under investigation. Its log takes the form $\frac{1}{\sqrt{n}} \sum_{i=1}^n \log(eU_i)$. Since $V_i := \log(eU_i) = \log(e) + \log(U_i) = 1 + \log(U_i)$ and $-\log(U_i)$ has an Exponential(1) distribution, the random variable $V_i = 1 + \log(U_i)$ has mean 0 and variance 1, so by the central limit theorem $\log(Y) \sim N(0,1)$ and has limiting probability density function $\frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\}$. Making a change of variables we see that the limiting pdf of Y is $\frac{1}{\sqrt{2\pi}y} \exp\{-\frac{1}{2}\log(y)^2\}$.

15. Let N be a random variable whose distribution is Poisson with parameter $\lambda > 0$, so that

$$P[N = n] = e^{-\lambda}\lambda^n/n!, \; n = 0, 1, 2, \ldots.$$

and suppose $X_1, X_2, \ldots$ is a sequence of iid Bernoulli random variables with parameter p, and independent of N. Thus

$$P[X_i = 1] = 1 - P[X_i = 0] = p, \; i = 1, 2, \ldots.$$

Define

$$S = \begin{cases} \sum_{i=1}^N X_i & \text{if } N > 0 \\ 0 & \text{if } N = 0 \end{cases}$$

Find a simple expression for the variance of S.

Solution: Write $Var(S) = Var(E(S|N)) + E(Var(S|N))$. Then we have $E(S|N) = NE(X_i) = Np$, and $Var(S|N) = NVar(X_i) = Np(1-p)$. Also, and $E(N) = Var(N) = \lambda$. This gives

$$Var(S) = Var(Np) + E[Np(1-p)] = \lambda p^2 + \lambda p(1-p) = p\lambda.$$