Department of Applied Mathematics and Statistics
The Johns Hopkins University

Introductory Examination–Spring Session

Tuesday, January 12, 2010

Instructions: Read carefully!

1. This **closed-book** examination consists of 15 problems, each worth 5 points. The passing grade has been set at 50 points, i.e. 2/3 of the total points. Partial credit will be given as appropriate; each part of a problem will be given the same weight. If you are unable to prove a result asserted in one part of a problem, you may still use that result to help in answering a later part.

2. You have been provided with a syllabus indicating the scope of the exam. Our purpose is to test not only your knowledge, but also your ability to apply that knowledge, and to provide mathematical arguments presented in **clear, logically justified steps**. The grading will reflect that broader purpose.

3. The problems have not been grouped by topic, but there are roughly equally many mainly motivated by each of the three areas identified in the syllabus (linear algebra; real analysis; probability;). Nor have the problems been arranged systematically by difficulty. If a problem directs you to use a particular method of analysis, you *must* use it in order to receive substantial credit.

4. Start your answer to each problem on a NEW sheet of paper. Write only on ONE SIDE of each sheet, and please do not write very near the margins on any sheet. Arrange the sheets in order, and write your NAME and the PROBLEM NUMBER on each sheet.

5. The examination will begin at 8:30 AM; lunch and refreshments will be provided. The exam will end just before 5:00 PM. You may leave before then, but in that case you may not return.

6. Paper will be provided, but you should bring and use writing instruments that yield marks dark enough to be read easily.

7. **No calculators of any sort are needed or permitted.**

1. A complex number $z \in \mathbb{C}$ is said to be *algebraic* if there are $k+1$ integers $n_0, ..., n_k$ with $n_k \neq 0$ such that
$$n_k z^k + \cdots + n_1 z + n_0 = 0.$$
Prove that the set of algebraic numbers in the real interval $[0,1]$ forms a dense subset of $[0,1]$ but that there are uncountably many non-algebraic numbers in $[0,1]$.

2. Fix an arbitrary $r \in \mathbb{N}$. Let p_n denote the probability when a fair coin is tossed (independently) n times that a run of r consecutive heads never appears. Observe that $p_0 = p_1 = \cdots = p_{r-1} = 1$. Prove that
$$p_n = \sum_{k=1}^{r} 2^{-k} p_{n-k}, \quad n \geq r.$$

3. Let $g(x) > 0$ be a function on $[0, \infty)$ for which
$$\int_0^\infty x g(x) dx < \infty$$
and define
$$a_n = \int_n^\infty g(x) dx, \ \ n = 1, 2, ...$$
Determine the convergence or divergence of $\sum_n a_n$.

4. Let S and T be two subsets in $\mathbb{R}^n$ defined as follows:
$$S = \{\ x\ : x_1 = x_2 = \cdots = x_n\}; \quad T = \{\ x\ : x_1 + x_2 + \cdots + x_n = 0\}.$$
(a) Show that S and T are subspaces.
(b) Show that for any vector $z \in \mathbb{R}^n$ there exist unique pair (x, y) such that $x \in S$ and $y \in T$ and $z = x + y$.

5. Let $y \neq 0$ in $\mathbb{R}^n$, and define $A = yy^T$, an $n \times n$ matrix. Find all the eigenvalues of A and find the algebraic and geometric multiplicities of each eigenvalue.

6. Prove the following statement or give a counterexample to show it is false:

   Given an open interval $(a, b) \subset [0, +\infty)$ and a positive integer N there exists $C > 0$ such that for every $x > C$ we have $x \in (na, nb)$ for some integer $n \geq N$.

7. Show that for every $n \times n$ complex matrix A which is not proportional to the identity matrix there exists at least one n-vector x, $x \neq 0$,which is not an eigenvector.

8. Let f and g be continuous functions defined on the set of real numbers, i.e., $f, g : \mathbb{R} \to \mathbb{R}$. Suppose that f and g have the same zeros and their common set of zeros is finite. That is, the sets

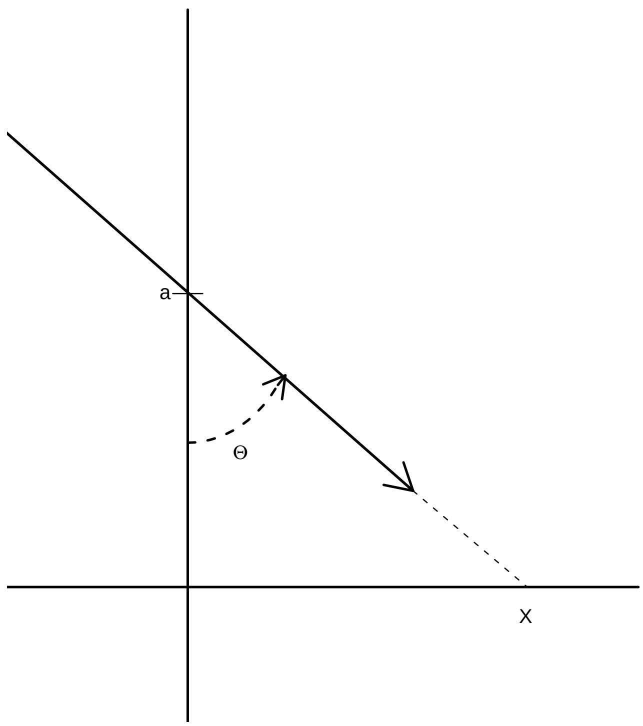
$$\{x \in \mathbb{R} \ : \ f(x) = 0\} \qquad \text{and} \qquad \{x \in \mathbb{R} \ : \ g(x) = 0\}$$

   are equal and finite.

   Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(a) = \lim_{x \to a} \frac{f(x)}{g(x)}.$$

   Prove or disprove: h is continuous.



9. An arrow whose center is located $a > 0$ units on the vertical axis is spun so that the angle it makes with the vertical axis is Θ, where Θ is a random variable uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (see figure). Determine the distribution of the x-coordinate that the arrow points to.

10. Suppose that X and Y are independent Geometric random variables with parameter p. Find the conditional distribution of X given $X + Y = n$, where $n \geq 1$.

11. An n by n matrix P is called a projection if $P^2 = P$. Prove that if P, Q and $P + Q$ are all projections, then $PQ = 0$.

12. If A and B are $n \times n$ real symmetric matrices, write $A \leq B$ if and only if $B - A$ is nonnegative definite.
Show that if A and B are $n \times n$ real symmetric positive definite matrices with $A \leq B$, then $B^{-1} \leq A^{-1}$.

13. The function $f : (0, 1) \to \mathbb{R}$ is continuous on the open interval $(0, 1)$, and satisfies $0 < f(x) < x$. Let a sequence of functions $f_n : (0, 1) \to \mathbb{R}$ be defined by

$$f_1(x) = f(x), \quad f_n(x) = f(f_{n-1}(x)), \quad \text{for } n \geq 2.$$

Prove that $f_n(x) \to 0$ as $n \to \infty$ for all $x \in (0, 1)$.

14. Suppose $U_1, \ldots, U_n$ are iid and distributed as Uniform$(0, 1)$. Find the probability density function of the limiting distribution of the random variable $\left(\prod_{i=1}^n eU_i\right)^{1/\sqrt{n}}$ as $n \to \infty$.

15. Let N be a random variable whose distribution is Poisson with parameter $\lambda > 0$, so that

$$P[N = n] = e^{-\lambda}\lambda^n/n!, \ n = 0, 1, 2, \ldots.$$

and suppose $X_1, X_2, \ldots$ is a sequence of iid Bernoulli random variables with parameter p, and independent of N. Thus

$$P[X_i = 1] = 1 - P[X_i = 0] = p, \ i = 1, 2, \ldots.$$

Define

$$S = \begin{cases} \sum_{i=1}^N X_i & \text{if } N > 0 \\ 0 & \text{if } N = 0 \end{cases}$$

Find a simple expression for the variance of S.